	GOVERNMENT ARTS AND SCIENCE COLLEG, KOVILPATTI – 628 503. (AFFILIATED TO MANONMANIAM SUNDARANAR UNIVERSITY, TIRUNELVELI) DEPARTMENT OF MATHEMATICS STUDY E - MATERIAL CLASS : II M.SC (MATHEMATICS) SEM: III SUBJECT : TOPOLOGY (PMAM32)

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Topology

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Text Book : **Topology (Second Edition),** James R Munkres, Prentice Hall of India Pvt. Ltd.

UNIT-I

0.1 Topological Spaces

Definition 0.1.1. A *topology* on a set X is a collection J of subsets of X having the following properties:

(i) $\varnothing$ and X are in J.

(ii) The union of the elements of any subcollection of J is in J.

(iii) The intersection of the elements of any finite subcollection of J is in J.

A set X for which a topology J has been specified is called a *topological space*.

If X is a topological space with topology J, we say that a subset U of X is an *open set* of X. If U belongs to the collection J.

If X is any set, the collection of all subsets of X is a topology on X, it is called the *discrete topology*. The collection consisting of X and $\varnothing$ only is also a topology on X, it is called the *indiscrete topology* or the *trivial topology*.

Let X be a set. Let J_f be a collection of all subsets U of X such that $X-U$ either is finite or is all of X. Then J_f is a topology on X, called the *finite complement topology*.

Result 0.1.2. J_f *is a finite complement topology.*

Proof. Since $X - X = \varnothing$ and $X - \varnothing = X$, either is finite or is all of X.

Both X and $\varnothing$ are in J_f.

To show that SU_α is in J_f.

$X - SU_\alpha = T(X - U_\alpha)$.

Since $X - U_\alpha$ is finite then $T(X - U_\alpha)$ is finite.

Then $(X - SU_\alpha)$ is finite.

Therefore, SU_α is in J_f.

If $U_1, U_2, \cdots, U_n$ or non empty elements of J_f.

To show that TU_i is in J_f.

Now we know that $X - T_{i=1}^{n} U_i = S_{i=1}^{n} (X - U_i)$.

since $(X - U_i)$ is finite then

$$\bigcup_{i=1}^{n} (X - U_i)$$ is finite.

Then $\bigcap U_\alpha$ is in $\mathcal{T}_f$.

Therefore, $\mathcal{T}_f$ is a finite complement topology.

Definition 0.1.3. Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a given set X. If $\mathcal{T}' \supset \mathcal{T}$, we say that $\mathcal{T}'$ is *finer* than $\mathcal{T}$; if $\mathcal{T}'$ properly contains $\mathcal{T}$, we say that $\mathcal{T}'$ is *strictly finer* than $\mathcal{T}$. We also say that $\mathcal{T}$ is *coarser* than $\mathcal{T}'$, or *strictly coarser*, in these two respective situations. We say $\mathcal{T}$ is *comparable* with $\mathcal{T}'$ if either $\mathcal{T}' \supset \mathcal{T}$ or $\mathcal{T} \supset \mathcal{T}'$.

0.2 Basis for a Topology

Definition 0.2.1. If X is a set, a *basis* for a topology on X is a collection $\mathcal{B}$ of subsets of X (called *basis elements*) such that

(i) For each $x \in X$, there is at least one basis element B containing x.

(ii) If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

If $\mathcal{B}$ satisfies these two conditions, then we define the *topology* $\mathcal{T}$ *generated by* $\mathcal{B}$ as follows: A subset U of X is said to be open in X (that is, to be an element of $\mathcal{T}$) if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$. Note that each basis element is itself an element of $\mathcal{T}$.

Lemma 0.2.2. *Let X be a set; let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathcal{B}$.*

Proof. Let X be a set and $\mathcal{B}$ be the basis for the topology $\mathcal{T}$ on X.

The collection of elements of $\mathcal{B}$ are also elements of $\mathcal{T}$ because $\mathcal{T}$ is a topology, their union is in $\mathcal{T}$.

Conversely, given $U \in \mathcal{T}$, choose for each $x \in U$ an element B_x of $\mathcal{B}$ such that $x \in B_x \subset U$. Then $U = \bigcup_{x \in U} B_x$, so U equals a union of elements of $\mathcal{B}$.

Lemma 0.2.3. *Let X be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X and each x in U, there is an element C of $\mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology of X.*

Proof. First we prove that $\mathcal{C}$ is a basis.

Given $x \in X$, since X is an open set, by hypothesis an element C of $\mathcal{C}$ such that $x \in C \subset X$.

Let $x \in C_1 \cap C_2$ where C_1 and C_2 are the elements of $\mathcal{C}$.

Since C_1 and C_2 are open, $C_1 \cap C_2$ are open.

By hyphothesis, there exists an element C_3 of $\mathcal{C}$ such that $x \in C_3 \subset C_1 \cap C_2$.

Therefore, $\mathcal{C}$ is a basis.

Let $\mathcal{J}$ be the topology on X.

Let $\mathcal{J}'$ denote the topology generated by $\mathcal{C}$.

To prove that $\mathcal{J}' = \mathcal{J}$.

By 0.2.4, $\mathcal{J}'$ is finer than $\mathcal{J}$.

Conversely, since each element of $\mathcal{C}$ is an element of $\mathcal{J}$, the union of elements of $\mathcal{C}$ is also in $\mathcal{J}$.

By 0.2.2, $\mathcal{J}'$ contains $\mathcal{J}$.

Therefore, $\mathcal{J}' = \mathcal{J}$.

Therefore, $\mathcal{C}$ is a basis for the topology of X.

Lemma 0.2.4. *Let* $\mathcal{B}$ *and* $\mathcal{B}'$ *be bases for the topologies* $\mathcal{J}$ *and* $\mathcal{J}'$*, respectively, on* X*. Then the following are equivalent:*

(i) $\mathcal{J}'$ *is finer than* $\mathcal{J}$*.*

(ii) For each $x \in X$ *and each basis element* $B \in \mathcal{B}$ *containing* x*, there is a basis element* $B' \in \mathcal{B}'$ *such that* $x \in B' \subset B$*.*

Proof. To prove (ii)⇒(i)

Given an element $U \in \mathcal{J}$.

To show that $U \in \mathcal{J}'$.

Let $x \in U$. Since $\mathcal{B}$ generates $\mathcal{J}$, there is an element $B \in \mathcal{B}$ such that $x \in B \subset U$.

By (ii), there exists an element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$, then $x \in B' \subset U$.

By definition of basis for the topology, $U \in \mathcal{J}'$.

To prove (i)⇒(ii)

Given $x \in X$ and $B \in \mathcal{B}$ with $x \in B$.

Now $B \in \mathcal{J}$, by definition and $\mathcal{J} \subset \mathcal{J}'$ by (i); therefore $B \in \mathcal{J}'$.

Since J' is generated by B', there is an element $B' \in B'$ such that $x \in B' \subset B$.

Definition 0.2.5. If B is the collection of all open intervals in the real line,

$(a, b) = \{x|a < x < b\}$,

the topology generated by B is called the *standard topology* on the real line. If B' is the collection of all half-open intervals of the form

$[a, b) = \{x|a \le x < b\}$,

where $a < b$, the topology generated by B' is called the *lower limit topology* on R. When R is given the lower limit topology, we denote it by R_l. Finally let K denote the set of all numbers of the form $1/n$, for $n \in Z_+$, and let B'' be the collection of all open intervals (a, b), along with all sets of the form $(a, b) - K$. The topology generated by B'' will be called the *K-topology* on R. When R is given this topology, we denote it by R_k.

Lemma 0.2.6. *The topologies of* R_l *and* R_k *are strictly finer than three standard topology on* R*, but are not comparable with one another.*

Proof. Let J, J', J'' be the topologies of R, R_l, R_k,respectively.

Given a basis element (a, b) for J and a point x of (a, b), the basis element $[x, b)$ for J' contains x and lies in (a, b). On the otherhand,given the basis element $[x, d)$ for J', there is no open interval (a, b) that contains x and lies in $[x, d)$. Thus J' is strictly finer than J.

Given a basis element (a, b) for J and a point x of (a, b), this same interval is a basis element for J'' that contains x. On the otherhand, given the basis element $B = (-1, 1) - K$ for J'' and the point 0 of B, there is no open interval that contains 0 and lies in B.

By definition of comparable, J' and J'' are not comparable with one another. 2

Definition 0.2.7. *A subbasis* S *for a topology on* X *is a collection of subsets of* X *whose union equals* X*. The topology generated by the subbasis* S *is defined to be the collection* J *of all unions of finite intersections of elements of* S*.*

0.3 The Order Topology

Definition 0.3.1. If X is a simply ordered set, there is a standard topology for X, defined using the order relation. It is called the *order topology*.
Suppose that X is a set having a simple order relation $<$. Given elements a and b of X such that $a < b$, there are four subsets of X that are called the *intervals* determined by a and b. They are the following:

$(a, b) = \{x|a < x < b\}$,

$(a, b] = \{x|a < x \leq b\}$,

$[a, b) = \{x|a \leq x < b\}$,

$[a, b] = \{x|a \leq x \leq b\}$.

A set of the first type is called an *open interval* in X, a set of the last type is called a *closed interval* in X, and sets of the second and third types are called *half-open intervals*.
Definition 0.3.2. Let X be a set with a simple order relation; assume X has more than one element. Let B be the collection of all sets of the following types:
(1) All open intervals (a, b) in X.
(2) All intervals of the form $[a_0, b)$, where a_0 is the smallest element(if any) of X.
(3) All intervals of the form $(a, b_0]$, where b_0 is the largest element(if any) of X.
The collection B is a basis for a topology on X, which is called the *order topology*.
Definition 0.3.3. If X is an ordered set, and a is an element of X, there are four subsets of X that are called *rays* determined by a. They are the following:

$(a, +\infty) = \{x|x > a\}$,

$(-\infty, a) = \{x|x < a\}$,

$[a, +\infty) = \{x|x \geq a\}$,

$(-\infty, a] = \{x|x \leq a\}$.

Sets of the first types are called *open rays*, and sets of the last two types are called *closed rays*.

0.5 The Subspace Topology

Definition 0.5.1. Let X be a topological space with topology J . If Y is a subset of X, the collection

$J_Y = \{Y \cap U|U \in J\}$

is a topology on Y , called the *subspace topology*. With this topology, Y is called a *subspace* of X; its open sets consist of all intersections of open sets of X with Y .
Lemma 0.5.2. *If* B *is a basis for the topology of* X *then the collection*

$\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$

is a basis for the subspace topology on Y *.*

Proof. Consider U is open in X. Given $\mathcal{B}$ is a basis for the topology of X. We can choose an element B of $\mathcal{B}$ such that $y \in B \subset U$.

Then $y \in B \cap Y \subset U \cap Y$, since $\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$.

By 0.2.3 or definition of basis, $\mathcal{B}_Y$ is a basis for the subspace topology on Y .

Definition 0.5.3. If Y is a subspace of X, we say that a set U is *open in* Y (or open *relative* to Y) if it belongs to the topology of Y ; this implies in particular that it is a subset of Y . We say that U is *open in* X if it belongs to the topology of X.

Lemma 0.5.4. *Let* Y *be a subspace of* X*. If* U *is open in* Y *and* Y *is open in* X*, then* U *is open in* X*.*

Proof. Given U is open in Y and Y is open in X.

Since U is open in Y and Y is a subspace of X then $U = Y \cap V$ where V is open in X.

Since Y and V are both open in X, $Y \cap V$ is open in X.

Therefore, U is open in X. 2

Theorem 0.5.5. *If* A *is a subspace of* X *and* B *is a subspace of* Y *, then the product topology on* $A \times B$ *is the same as the topology* $A \times B$ *inherits as a subspace of* $X \times Y$ *.*

Proof. The set $U \times V$ is the general basis element for $X \times Y$, where U is open in X and V is open in Y .

Then $(U \times V) \cap (A \times B)$ is the general basis element for the subspace topology on $A \times B$. Now

$(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B)$.

Since $U \cap A$ and $V \cap B$ are the general open sets for the subspace topologies on A and B respectively, the set $(U \cap A) \times (V \cap B)$ is the general basis element for the product on $A \times B$.

The bases for the subspace topology on $A \times B$ and for the product topology on $A \times B$ are the same. Hence the topologies are the same.

Theorem 0.5.6. *Let* X *be an ordered set in the order topology; let* Y *be a subset of* X *that is convex in* X*. Then the order topology on* Y *is the same as the topology* Y *inherits as a subspace of* X*.*

Proof. Consider the ray $(a, +\infty)$ in X.

If $a \in Y$, then $(a, +\infty) \cap Y = \{x \mid x \in Y \text{ and } x > a\}$; this is an open ray of the ordered set Y.

If $a \notin Y$, then a is either a lower bound on Y or an upper bound on Y, since Y is convex.

If $a \in Y$, the set $(a, +\infty) \cap Y$ equals all of Y. If $a \notin Y$, it is empty.

Similarly the intersection of the ray $(-\infty, a) \cap Y$ is either an open ray of Y, or Y itself or empty.

Since the sets $(a, +\infty) \cap Y$ and $(-\infty, a) \cap Y$ form a subbasis for the subspace topology on Y and since each is open in the order topology, the order topology
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contains the subspace topology.

Conversely, Y equals the intersection of X with Y, that is $X \cap Y = Y$. So it is open in the subspace topology on Y. The order topology is contained in the subspace topology. Therefore, the order topology and subspace topology are same.

0.6 Closed Sets and Limit Points

Definition 0.6.1. A subset A of a topological space X is said to be *closed* if the set $X - A$ is open.

Theorem 0.6.2. *Let X be a topological space. Then the following conditions hold:*

(1) $\varnothing$ and X are closed.

(2) Arbitrary intersections of closed sets are closed.

(3) Finite unions of closed sets are closed.

Proof. (1) $\varnothing$ and X are closed because they are the complements of the open set X and $\varnothing$ respectively.

(2) Consider a collection of closed sets $\{A_\alpha\}_{\alpha\in J}$, we apply De Morgan's law,

$$X - \bigcap_{\alpha\in J} A_\alpha = \bigcup_{\alpha\in J} (X - A_\alpha)$$

Since the sets $X - A_\alpha$ are open. By definition of closed sets, the right side of this equation represents an arbitrary union of open sets and is thus open. Therefore, $\bigcap A_\alpha$ is closed.

(3) Similarly, if A_i is closed for $i = 1, 2, \cdots, n$. Consider the equation

$$X - \bigcup_{i=1}^{n} A_i = \bigcap_{i=1}^{n}$$

$(X - A_i)$
The set on the right side of this equation is a finte intersection of open sets and is therefore open. Hence SA_i is closed.

Definition 0.6.3. If Y is a subspace of X, we say that a set A is *closed in* Y if A is a subset of Y and if A is closed in the subspace topology of Y (that is, if $Y - A$ is open in Y).

Theorem 0.6.4. *Let* Y *be a subspace of* X*. Then a set* A *is closed in* Y *if and only if it equals the intersection of a closed set of* X *with* Y*.*

Proof. Assume that $A = C \cap Y$, where C is closed in X. Then $X - C$ is open in X, so that $(X - C) \cap Y$ is open in Y. By the definition of the subspace topology, but $(X - C) \cap Y = Y - A$. Hence $Y - A$ is open in Y, so that A is closed in Y.
Conversely, assume that A is closed in Y. Then $Y - A$ is open in Y. By definition, it equals the intersection of an open set U of X with Y. The set $X - U$ is closed in X and $A = Y \cap (X - U)$. Hence A equals the intersection of a closed set of X with Y.

Theorem 0.6.5. *Let* Y *be a subspace of* X*. If* A *is closed in* Y *and* Y *is closed in* X*, then* A *is closed in* X*.*

Proof. Given A is closed in Y and Y is closed in X. Since A is closed in Y and Y is a subspace of X.
Let $A = Y \cap (X - B)$ where $X - B$ is open in X. Then B is closed in X. Since Y and B are both closed in X. Then $Y \cap (X - B)$ is closed in X. Therefore, A is closed in X.

Definition 0.6.6. Given a subset A of a topological space X, the *interior* of A is defined as the union of all open sets contained in A, and the *closure* of A is defined as the intersection of all closed sets containing A.
The interior of A is denoted by Int A and the closure of A is denoted by Cl A or by A. Obviously Int A is an open set and A is a closed set; furthermore,
Int $A \subset A \subset A$.
If A is open, A=Int A; while if A is closed, $A = A$.

Theorem 0.6.7. *Let* Y *be a subspace of* X*; let* A *be a subset of* Y*; let* A *denote the closure of* A *in* X*. Then the closure of* A *in* Y *equals* $A \cap Y$*.*

Proof. Let B denote the closure of A in Y. The set A is closed in X, so $A \cap Y$ is closed in Y. By 0.6.4, since $A \cap Y$ contains A and since B is closed. By definition B equals the intersection of all closed subsets of Y containing A, we must have $B \cap (A \cap Y)$.
On the otherhand, we know that B is closed in Y. By 0.6.4, $B = C \cap Y$ for some set C closed in X. Then C is a closed set of X containing A; because

A is the intersection of all such closed sets, we conclude that $A \subset C$. Then

$(A \cap Y) \subset (C \cap Y) = B$. Therefore, $B = A \cap Y$. 2

Theorem 0.6.8. *Let A be a subset of the topological space X.*

(a) Then $x \in A$ if and only if every open set U containing x intersects A.

(b) Supposing the topology of X is given by a basis, then $x \in A$ if and only if every basis element B containing x intersects A.

Proof. (a)We prove this theorem by contrapositive method.
If x is not in A, since A is closed, $A = A$. The set $U = X - A$ is an open set containing x that does not intersect A.
Conversely, if there exists an open set U containing x which does not intersect A. Then $X - U$ is a closed set containing A.

By definition of the closure A, the set $X - U$ must contain A, since $x \in U$.

Therefore, x cannot be in A.
(b) Write the definition of topology generated by basis,if every open set x intersects A, so does every basis element B containing x, because B is an open set.
Conversely, if every basis element containing x intersects A, so does every open set U containing x, because U contains a basis element that contains x.
Definition 0.6.9. If A is a subset of the topological space X and if x is a point of X, we say that x is a *limit point*(or "cluster point" or "point of accumulation") of A if every neighborhood of x intersects A in some point other than x itself.

Said differently, x is a limit point of A if it belongs to the closure of $A - \{x\}$.

The point x may lie in A or not; for this definition it does not matter.
Theorem 0.6.10. *Let A be a subset of the topological space X; let A' be the set of all limit points of A. Then $A = A \cup A'$.*

Proof. Let A' be the set of all limit points of A.

If $x \in A'$, every neighborhood of x intersects of A in a point different from x. By 0.6.8, $x \in A$. Then $A' \subset A$.

By definition of closure, $A \subset A$. Therefore, $A \cup A' \subset A$.

Conversely, let $x \in A$

To show that $A \subset A \cup A'$

If $x \in A$ then it is trivially true for $x \in A \cup A'$.

Suppose $x \notin A$. Since $x \in A$, by 0.6.8, we know that every neighborhood U of x intersect A, because $x \notin A$, the set U must intersect A in a point different from x. Then $x \in A'$ so that $x \in A \cup A'$.

Then $A \subset A \cup A'$.

Therefore, $A = A \cup A'$.

Corollary 0.6.11. *A subset of a topological space is closed if and only if it contains all its limit points.*

Proof. The set A is closed iff $A = A$. By 0.6.10, $A' \subset A$.

Definition 0.6.12. A topological space X is called a *Hausdroff space* if for each pair x_1, x_2 of distinct points of X, there exist neighborhoods U_1 and U_2 of x_1 and x_2 respectively, that are disjoint.

Theorem 0.6.13. *Every finite point set in a Hausdorff space X is closed.*

Proof. It is enough to show that every one-point set $\{x_0\}$ is closed.

If x is a point of X different from x_0, then x and x_0 have disjoint neighborhoods U and V respectively.

Since U does not intersect $\{x_0\}$, the point x cannot belong to the closure of the set $\{x_0\}$.

As a result, the closure of the set $\{x_0\}$ is $\{x_0\}$ itself.

Therefore, $\{x_0\}$ is closed.

Note: The condition that finite point sets be closed is in fact weaker than the Hausdroff condition. For example, the real line R in the finite complement topology is not a Hausdorff space, but it is a space in which finite point sets are closed. The condition that finite point sets be closed has been given a name of its own; it is called the T_1 *axiom*.

Theorem 0.6.14. *Let X be a space satisfying the T_1 axiom; let A be a subset of X. Then the point x is a limit point of A if and only if every neighborhood of x contains infinitely many points of A.*

Proof. If every neighborhood of x intersects A in infinitely many points, it certainly intersects A in some point other than x itself, so that x is a limit point of A.

Conversely, suppose that x is a limit point of A and suppose some neighborhood U of x intersects A in only finitely many points.

Let $\{x_1, x_2, \cdots, x_m\}$ be the points of $U \cap (A - \{x\})$.

The set $X - \{x_1, x_2, \cdots, x_m\}$ is an open set of X, since the finite point set $\{x_1, x_2, \cdots, x_m\}$ is closed then

$U \cap (X - \{x_1, x_2, \cdots, x_m\})$

is a neighborhood of x that does not intersects the set $A-\{x\}$. Since $\{x_1, x_2, \cdots, x_m\}$ be points of $U \cap (A - \{x\})$.

This contradicts the assumption that x is a limit point of A.

Theorem 0.6.15. *If X is a Hausdorff space, then a sequence of points of X converges to at most one point of X.*

Proof. Suppose that x_n is a sequence of points of X that converges to x.

If $y \neq x$, let U and V be disjoint neighborhoods of x and y respectively. Since U contains x_n for all but finitely many values of n, the set V cannot contains x_n. Therefore, x_n cannot converge.

If the sequence x_n of points of the Hausdorff space X converges to the point x of X, we often write $x_n \to x$.

Therefore, x is the limit of the sequence x_n.

Theorem 0.6.16. *Every simply ordered set is a Hausdorff space in the order topology. The product of two Hausdorff spaces is a Hausdorff space. A subspace of a Hausdorff space is a Hausdorff space.*

Proof. Let X and Y be two Hausdorff spaces.

To prove $X \times Y$ is Hausdorff.

Let $x_1 \times y_1$ and $x_2 \times y_2$ be two distinct points of $X \times Y$. Then x_1, x_2 are distinct points of X and X is a Hausdorff space, there exists neighborhood U_1 and U_2 of x_1 and x_2 such that $U_1 \cap U_2 = \varnothing$

Similarly, y_1, y_2 are distinct point of Y and Y is a Hausdorff space, there exists neighborhood V_1 and V_2 of y_1 and y_2 such that $V_1 \cap V_2 = \varnothing$.

Then clearly $U_1 \times V_1$ and $U_2 \times V_2$ are open sets in $X \times Y$ containing $x_1 \times y_1$ and $x_2 \times y_2$ such that $(U_1 \times V_1) \cap (U_2 \times V_2) = \varnothing$.

Therefore, $X \times Y$ is a Hausdorff space.

Let X be a Hausdorff space and let Y be a subspace.

To prove Y is a Hausdorff space.

Let y_1, y_2 be two distinct points of Y and Y containing X. Then y_1 and y_2 are distinct points in X and X is Hausdorff there exists neighborhood U_1 and U_2 of y_1 and y_2 such that $U_1 \cap U_2 = \varnothing$. Then $U_1 \cap Y$ and $U_2 \cap Y$ are distinct neighborhoods of y_1 and y_2 in Y .

Therefore, Y is a Hausdorff space.

UNIT-II

0.4 The product Topology on $X \times Y$

Definition 0.4.1. Let X and Y be topological spaces. The *product topology* on $X \times Y$ is the topology having as basis the collection $\mathcal{B}$ of all sets of the form $U \times V$, where U is an open subset of X and V is an open subset of Y.

Theorem 0.4.2. *If* $\mathcal{B}$ *is a basis for the topology of* X *and* $\mathcal{C}$ *is a basis for the topology of* Y*, then the collection*

$$\mathcal{D} = \{B \times C \mid B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}$$

is a basis for the topology of $X \times Y$.

Proof. We apply 0.2.3. Given an open set W of $X \times Y$ and a point $x \times y$ of W, by definition of the product topology there is a basis element $U \times V$ such that

$$x \times y \in U \times V \subset W.$$

Because $\mathcal{B}$ and $\mathcal{C}$ are bases for X and Y respectively, we can choose an element B of $\mathcal{B}$ such that $x \in B \subset U$ and an element C of $\mathcal{C}$ such that $y \in C \subset V$. Then

$$x \times y \in B \times C \subset W.$$

Therefore, $\mathcal{D}$ is a basis for $X \times Y$.

Definition 0.4.3. Let $\pi_1 : X \times Y \to X$ be defined by the equation

$$\pi_1(x, y) = x;$$

let $\pi_2 : X \times Y \to Y$ be defined by the equation

$$\pi_2(x, y) = y.$$

The maps π_1 and π_2 are called the *projections* of $X \times Y$ onto its first and second factors, respectively.

We use the word "onto" because π_1 and π_2 are surjective.

Note If U is an open subset of X, then the set $\pi_1^{-1}(U)$ is precisely the set $U \times Y$, which is open in $X \times Y$. Similarly, if V is open in Y, then

$$\pi_2^{-1}(V) = X \times V,$$

which is also open in $X \times Y$. The intersection of these two sets is the set $U \times V$.

Theorem 0.4.4. *The collection*

$$\mathcal{S} = \{\pi_1^{-1}(U) \mid U \text{ open in } X\} \cup \{\pi_1^{-1}$$

$_2(V)|V$ *open in* $Y\}$

is a subbasis for the product topology on $X \times Y$ *.*

Proof. Let J denote the product topology on $X \times Y$.

Let J' be the topology generated by S. Because every element of S belongs to J .

By definition of subbasis, arbitrary unions of finite intersections of elements of S.

Thus $J' \subset J$.
On the otherhand,
$U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V)$
where $\pi_1^{-1}(U)$ is open in X and $\pi_2^{-1}(V)$ is open in Y .
Since $U \times V \in J$, we have $U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V)$. $U \times V \in J'$. Therefore,

$J \subset J'$.

0.7 Continuity of a Function

Definition 0.7.1. Let X and Y be topological spaces. A function $f : X \to Y$ is said to be *continuous* if for each open subset V of Y , the set $f^{-1}(V)$ is an open subset of X.

$f^{-1}(V)$ is the set of all points x of X for which $f(x) \in V$; it is empty if V does not intersect the image set $f(X)$ of f.

Theorem 0.7.2. *Let* X *and* Y *be the topological spaces.Let* $f : X \to Y$ *. Then the following are equivalent:*

(a) f *is continuous.*

(b) For every subset of X*, one has* $f(\bar{A}) \subset \overline{f(A)}$.

(c) For every closed set B *of* Y *, a set* $f^{-1}(B)$ *is closed in* X.

(d) For each $x \in X$ *and each neighborhood* V *of* $f(x)$ *there is a neighborhood* U *of* x *such that* $f(U) \subset V$ *.*

If the condition in equation (d) holds for the point x *of* X *such that* f *is continuous* *at the point* x.

Proof. To show that (a)$\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$(a) and (a)$\Rightarrow$ (d), (d) $\Rightarrow$ (a).

First we show that (a)$\Rightarrow$(b)

Assume f is continuous. Let A be a subset of X. We have to show that $f(A) \subset f(A)$.

If $x \in A$ then $f(x) \in f(A)$. Since f is continuous, $f^{-1}(V)$ is an open set of X containing x, where V be a neighborhood of $f(x)$.

Now $f^{-1}(V)$ must intersect A in some point y. Then V intersects $f(A)$ in the point $f(y)$, $f(x) \in f(A)$. Therefore, $f(A) \subset f(A)$.

To show that (b)$\Rightarrow$(C)

Let B be closed in Y. Let $A = f^{-1}(B)$.

To prove that A is closed in X.

ie, To prove that $A = A$.

By elementary set theory, we have $f(A) = f(f^{-1}(B)) \subset B$

If $x \in A$, then $f(x) \in f(A) \subset f(A) \subset B = B$.

Then $x \in f^{-1}(B) \Rightarrow x \in A$. Therefore, $A \subset A$.

Since $A \subset A$, therefore, $A = A$.

To show that (c)$\Rightarrow$(a)

Let V be open in Y. The set $B = Y - V$.

Then $f^{-1}(B) = f^{-1}(Y - V) = f^{-1}(Y) - f^{-1}(V) = X - f^{-1}(V)$

Now B is a closed set of Y then $f^{-1}(B)$ is closed in X(By hypothesis).

Then $f^{-1}(V)$ is open in X.

Therefore, f is continuous.

To show that (a)$\Rightarrow$(d)

Let $x \in X$. Let V be a neighborhood of $f(x)$. Then the set $U = f^{-1}(V)$ is a neighborhood of x.

Therefore, $f(U) \subset V$.

To show that (d)$\Rightarrow$(a)

Let V be open in Y. Let $x \in f^{-1}(V)$. Then $f(x) \in V$.

Then by hypothesis, there is a neighborhood U_x of x such that $f(U_x) \subset V$. Then $U_x \subset f^{-1}(V)$.

Now $f^{-1}(V)$ can be written as the union of the open sets U_x.

Thus $f^{-1}(V)$ is open.

Therefore, f is continuous.

Definition 0.7.3. Let X and Y be topological spaces. Let $f : X \to Y$ be a

bijection. If both the function x and the inverse function $f^{-1}(V)$ are continuous then f is called homeomorphism.
Theorem 0.7.4. *(Rules for constructing continuous functions). Let* X, Y *and* Z *be topological spaces.*

(a) (constant function) If $f : X \to Y$ *maps all of* X *into the single point* y_0 *of* Y*, then* f *is continuous.*

(b) (Inclusion) If A *is a subspace of* X*, the inclusion function* $j : A \to X$ *is continuous.*

(c) (Composites) If $f : X \to Y$ *and* $g : Y \to Z$ *are continuous, then the map* $g \circ f : X \to Z$ *is continuous.*

(d) (Restricting the domain) If $f : X \to Y$ *is a continuous. Let* A *is a subspace of* X*. Then the restricted function* $f/A : A \to Y$ *is continuous.*

(e) (Restricting or expanding the range) Let $f : X \to Y$ *be a continuous. If* Z *is a subspace of* Y *containing the image set* $f(X)$*, then the function* $g : X \to Z$ *obtained by restricting the range of* f *is continuous.*
If Z *is a space having* Y *as a subspace then the function* $h : X \to Z$ *obtained by expanding the range of* f *is continuous.*

(f) (Local formulation of continuity) The map $f : X \to Y$ *is continuous, if* X *can be written as the union of open set* U_α *such that* f/U_α *is continuous for each* α*.*

Proof. (a) Let $f(x) = y_0$, $x \in X$, $y_0 \in Y$.

Let V be open in Y .

If $y_0 \in Y$, the set $f^{-1}(V) = X$.

The set $f^{-1}(V)$ be open in X, $y_0 \subset V$

Therefore, f is continuous.

(b) Let A be a subspace of X. To prove $j : A \to X$ is continuous.

If U is open in X then $j^{-1}(U) = U \cap A$ which is open in A by definition of subspace topology.
Then $j^{-1}(U)$ is open in A.
Therefore, j is continuous.
(c) Since f and g be continuous. We have the following conditions:
If U is open in Z then $g^{-1}(U)$ is open in Y and $f^{-1}(g^{-1}(U))$ is open in X. But $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$.

Then $(g \circ f)^{-1}(U)$ is open in X. Therefore, $g \circ f : X \to Z$ is continuous.

(d)Let $f : X \to Y$ be continuous. Let A be a subspace of X.

To prove $f/A : A \to Y$ is continuous.

Since by (b), we have the inclusion map $j : A \to X$ is continuous. Also we have $f : X \to Y$ is continuous.

Therefore, the restricted function $f/A : A \to Y$ is continuous by (c).
ie, f/A each equals the composite of the inclusion map j.

(e) Let $f : X \to Y$ is continuous.

Given Z is a subspace of Y containing the image set $f(X)$. ie,$f(X) \subset Z \subset Y$

To prove the function $g : X \to Z$ obtained from f is continuous.

Let B be open in Z. Since Z is a subspace of Y, $B = Z \cap U$ for some open set U of Y .

Since B is open in Z, $g^{-1}(B)$ is open in X and since U is open in Y , $f^{-1}(U)$ is open in X

Then $f^{-1}(U) = g^{-1}(B)$

Therefore, $g : X \to Z$ obtained from f is continuous.

If Z is a space having Y as a subspace. To prove the function $h : X \to Z$ is continuous.

This is obtained by the composition of the map $f : X \to Y$ and the inclusion map $j : Y \to Z$.

Since Y is a subspace of Z, inclusion map $j : Y \to Z$ is continuous by (b).

Therefore, the function $h : X \to Z$ is continuous.

(f) Given X can be written as the union of open sets U_α such that f/U_α is continuous for each α .

To prove $f : X \to Y$ is continuous.

Let V be open in Y .

Now $f(x) \in V$, $x \in X$. Since U_α is open in X containing x. Then $f^{-1}(V) \cap U_\alpha$ is open in X.

Since f/U_α is continuous; U_α is open in X, $(f/U_\alpha)^{-1}(V)$ is open in X.

Then $f^{-1}(V)$ is open in X.

Therefore, f is continuous.

Theorem 0.7.5. *(The Pasting Lemma) Let* $X = A \cup B$*, where* A *and* B *are closed in* X*. Let* $f : A \to Y$ *and* $g : B \to Y$, B *is continuous. If* $f(x) = g(x)$ *for every* $x \in A \cap B$*, then* f *and* g *combine to give a continuous function* $h : X \to Y$ *defined by setting* $h(x) = f(x)$ *if* $x \in A$ *and* $h(x) = g(x)$ *if* $x \in B$.

Proof. Let $X = A \cup B$ where A and B are closed in X.

Since $f : A \to Y$ is continuous, $f^{-1}(C)$ is closed in A, where C is closed in Y .

Since $g : B \to Y$ is continuous, $g^{-1}(C)$ is closed in B where C is closed in Y .

If $x \in A$, $h(x) = f(x)$ and if $x \in B$, $h(x) = g(x)$.

If $x \in A \cup B$, $h(x) = f(x) \cup g(x)$.

Now $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$.

Then $h^{-1}(C)$ is closed in $A \cup B$.

Then $h^{-1}(C)$ is closed in X.

Therefore, h is continuous.

Theorem 0.7.6. *(Maps into products) Let* $f : A \to X \times Y$ *be given by the equation*

$f(a) = (f_1(a), f_2(a))$.

Then f *is continuous if and only if the functions*

$f_1 : A \to X$ *and* $f_2 : A \to Y$

are continuous.

The maps f_1 *and* f_2 *are called the coordinate functions of* f*.*

Proof. Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be projections onto its first and second factors. These maps are continuous..

For, $\pi_1^{-1}(U) = U \times Y$ and $\pi_2^{-1}(V) = X \times V$.

If U and V are open, these sets are open.

Since $f : A \to X \times Y$, $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$, for every $a \in A$.

Since $f_1 : A \to X$ and $f_2 : A \to Y$

$f_1(a) = \pi_1(f(a))$ and $f_2(a) = \pi_2(f(a))$

If the function f is continuous, then f_1 and f_2 are composites of continuous functions, f_1 and f_2 are continuous.

Conversely, suppose f_1 and f_2 are continuous. Then $f_1^{-1}(U)$ is open in A and

$f_2^{-1}(V)$ is open in A.

$a \in f_1^{-1}(U) \cap f_2^{-1}(V)$

Also we have $U \times V$ be the basis element for the topology on $X \times Y$ then

$f(a) \in U \times V$

$\Rightarrow a \in f^{-1}(U \times V)$

$\Rightarrow f_1^{-1}(U) \cap f_2^{-1}(V) \subset f^{-1}(U \times V)$

Also if $a \in f^{-1}(U \times V) \Rightarrow f(a) \in U \times V$

$\Rightarrow (f_1(a),\ f_2(a)) \in U \times V$

$\Rightarrow f_1(a) \in U,\ f_2(a) \in V$

$\Rightarrow a \in f_1^{-1}(U),\ a \in f_2^{-1}(V)$

$f^{-1}(U \times V) \subset f_1^{-1}(U) \cap f_2^{-1}(V)$

$f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V)$

Since $f_1^{-1}(U)$ and $f_2^{-1}(V)$ is open in A.

Then $f_1^{-1}(U) \cap f_2^{-1}(V)$ is open in A.

Then $f^{-1}(U \times V)$ is open in A.

Therefore, f is continuous.

0.8 The Product Topology

Definition 0.8.1. Let J be an index set. Given a set X, we define *J-tuple* of elements of X to be a function $x : J \to X$. If α is an element of J, we often denote the value of x at α by x_α rather than $x(\alpha)$; we call it the αth *coordinate* of x. And we often denote the function x itself by the symbol

$(x_\alpha)_{\alpha \in J}$,

which is as close as we can come to a tuple notation for an arbitrary index set J. We denote the set of all J-tuples of elements of X by X^J.

Definition 0.8.2. Let $\{A_\alpha\}_{\alpha \in J}$ be an indexed family of sets; let $X = \bigcup_{\alpha \in J} A_\alpha$. The *cartesian product* of this indexed family, denoted by

$\prod_{\alpha \in J}$

$$A_\alpha,$$

is defined to be the set of all J-tuples $(x_\alpha)_{\alpha\in J}$ of elements of X such that $x_\alpha \in A_\alpha$ for each $\alpha \in J$. That is, it is the set of all functions

$$x : J \to \bigcup_{\alpha\in J} A_\alpha$$

such that $x(\alpha) \in A_\alpha$ for each $\alpha \in J$.

Definition 0.8.3. Let $\{X_\alpha\}_{\alpha\in J}$ be an indexed family of topological spaces. Let us take as a basis for a topology on the product space

$$\prod_{\alpha\in J} X_\alpha,$$

the collection of all sets of the form

$$\prod_{\alpha\in J} U_\alpha,$$

where U_α is open in X_α, for each $\alpha \in J$. The topology generated by this basis is called the *box topology*.

Definition 0.8.4. Let

$$\pi_\beta : \prod_{\alpha\in J} X_\alpha \to X_\beta$$

be mapping is defined by

$$\pi_\beta((x_\alpha)_{\alpha\in J}) = x_\beta;$$

is called the *projection mapping* associated with the index β.

Definition 0.8.5. Let S_β denote the collection

$$S_\beta = \{\pi^{-1}_\beta(U_\beta) \mid U_\beta \text{ open in } X_\beta\},$$

and let S denote the union of these collections,

$$S = \bigcup_{\beta\in J} S_\beta.$$

The topology generated by the subbasis S is called the *product topology*. In this topology $\prod_{\alpha\in J} X_\alpha$ is called a *product space*.

Theorem 0.8.6. *(Comparison of the box and product topologies). The box topology on $\prod X_\alpha$ has as basis all sets of the form $\prod U_\alpha$, where U_α is open in X_α for each α. The product topology on $\prod X_\alpha$ has as basis all sets of the form $\prod U_\alpha$, where U_α is open in X_α for each α and U_α equals X_α except for finitely many values of α.*

Proof. By definition of box topology, the basis for box topology on $\prod X_\alpha$ is

$$B_b = \{\prod U_\alpha \mid U_\alpha \text{ is open in } X_\alpha\}.$$

By definition of product topology the basis for the topology on $\prod X_\alpha$ is B_p then B_p is the collection of all finite intersection of elements of S where $S = \bigcup_{\beta \in J} S_\beta$ and $S_\beta = \{\pi_\beta^{-1}(U_\beta) | U_\beta \text{ is open in } X_\beta\}$.

Case1:

We take finite intersection of elements of S_β.

Let $\pi_\beta^{-1}(U_\beta)$, $\pi_\beta^{-1}(V_\beta)$, $\pi_\beta^{-1}(W_\beta) \in S_\beta$.

Let $B = \pi_\beta^{-1}(U_\beta) \cap \pi_\beta^{-1}(V_\beta) \cap \pi_\beta^{-1}(W_\beta)$

$= \pi_\beta^{-1}(U_\beta \cap V_\beta \cap W_\beta) \in S_\beta \subset B_p$

$= \pi_\beta^{-1}(U'_\beta)$ where $U'_\beta = U_\beta \cap V_\beta \cap W_\beta$

$B = \prod_{\alpha \in J} U'_\alpha$ where U'_α is open in X_α, for $\alpha = \alpha_1, \alpha_2, \cdots, \alpha_n$ and $U'_\alpha = X_\alpha$ for $\alpha \neq \alpha_1, \alpha_2, \cdots, \alpha_n$.

Case 2:

We take intersection of elements from different S_β's.

Let $B' = \pi_\beta^{-1}(U_{\beta_1}) \cap \pi_\beta^{-1}(U_{\beta_2}) \cap \cdots \pi_\beta^{-1}(U_{\beta_n})$

$B' = \pi_\beta^{-1}(U_{\beta_1} \cap U_{\beta_2} \cap \cdots \cap U_{\beta_n})$

Let $x = (x_\alpha)_{\alpha \in J} \in B'$

Then $x = (x_\alpha)_{\alpha\in J} \in B' \Leftrightarrow (x_\alpha)_{\alpha\in J} \in \pi^{-1}_\beta(U_{\beta_1}) \cap \cdots \cap \pi^{-1}_\beta(U_{\beta_n})$

$\Leftrightarrow (x_\alpha)_{\alpha\in J} \in \cdots U_{\beta_1} \times \cdots \times U_{\beta_2} \times \cdots \times U_{\beta_n} \times \cdots$

$\Leftrightarrow x_\alpha \in U_\alpha$ for $\alpha = \beta_1, \beta_2, \cdots, \beta_n$ and $x_\alpha \in X_\alpha$ for $\alpha \neq \beta_1, \beta_2, \cdots, \beta_n$

$\Leftrightarrow (x_\alpha) \in \prod_{\alpha\in J} U_\alpha$ where U_α is open in X_α, for $\alpha = \beta_1, \beta_2, \cdots, \beta_n$ and $U_\alpha = X_\alpha$ for $\alpha \neq \beta_1, \beta_2, \cdots, \beta_n$

$B' = \prod_{\alpha\in J} U_\alpha$ where U_α is open in X_α.

Hence in both cases we get every basis element of the product topology in $\prod X_\alpha$ is of the form $\prod U_\alpha$ where U_α is open in X_α and $U_\alpha = X_\alpha$ except for finitely many values of α.

Clearly the basis $\mathcal{B}_p \subset \mathcal{B}_b$

Therefore, the box topology is finer than the product topology.

Theorem 0.8.7. *Suppose the topology on each space X_α is given by a basis $\mathcal{B}_\alpha$. The collection of all sets of the form*

$$\prod_{\alpha\in J} B_\alpha,$$

where $B_\alpha \in \mathcal{B}_\alpha$ for each α, will serve as a basis for the box topology on $\prod_{\alpha\in J} X_\alpha$.

The collection of all sets of the same form, where $B_\alpha \in \mathcal{B}_\alpha$ for finitely many indices α and $B_\alpha = X_\alpha$ for all the remaining indices, will serve as a basis for the product topology $\prod_{\alpha\in J} X_\alpha$.

Proof. Let $l = \{\prod_{\alpha\in J} B_\alpha \in \mathcal{B}_\alpha, \mathcal{B}_\alpha$ is a basis for $X_\alpha\}$ for each α.

B_α is a collection of open sets in X_α, for every α.

$\prod_{\alpha\in J} U_\alpha$ is open in $\prod_{\alpha\in J} X_\alpha$.

Therefore l is a collection of open sets in $\prod X_\alpha$.

To prove l is a basis for the box topology in $\prod_{\alpha\in J} X_\alpha$.

Now, $x = (x_\alpha)_{\alpha\in J} \in \prod_{\alpha\in J} X_\alpha$.

Let U be an open set in $\prod X_\alpha$ containing x.

Now U is an open set in the box topology in QX_{α}, $x \in U$, there exists a basis element $Q_{\alpha\in J}$

U_{α} such that $x \in Q_{\alpha\in J}$

$U_{\alpha} \subset U \Rightarrow x_{\alpha} \in U_{\alpha}$ for each α.

Now $x_{\alpha} \in U_{\alpha}$ and U_{α} is open in X_{α} and B_{α} is a basis for X_{α}, there exists $B_{\alpha} \in B_{\alpha}$

such that $x_{\alpha} \in B_{\alpha} \subset U_{\alpha}$ for each α.

Then $(x_{\alpha})_{\alpha\in J} \in Q_{\alpha\in J}$

$B_{\alpha} \subset Q_{\alpha\in J}$

$U_{\alpha} \subset U$.

ie, $x \in Q_{\alpha\in J}$

$B_{\alpha} \subset U$

For every $x \in QX_{\alpha}$ and any open set U containing x, there exists $Q_{\alpha\in J}$
B_{α} in 1

such that $x \in Q_{\alpha\in J}$

$B_{\alpha} \subset U$.

By 0.2.3, 1 is a basis for the box topology on the product space $Q_{\alpha\in J}$
X_{α}.

Let $1' = \{Q_{\alpha\in J}$

$B_{\alpha} | B_{\alpha}$, for finitely many indices and $B_{\alpha} = X_{\alpha}$ for the remaining

indices$\}$

To prove that $1'$ is a basis for the product topology on $Q_{\alpha\in J}$
X_{α}.

Let $x = (x_{\alpha}) \in Q_{\alpha\in J}$
X_{α}.

Let V be an open set in $Q_{\alpha\in J}$
X_{α} containing x, there exists a basis element $Q_{\alpha\in J}$
U_{α}
for the product topology in $Q_{\alpha\in J}$

X_{α} such that $x \in Q_{\alpha\in J}$

$U_{\alpha} \subset V$, where U_{α} is open

in X_{α} for $\alpha = \alpha_1, \alpha_2, \cdots, \alpha_n$ and $U_{\alpha} = X_{\alpha}$ for α 6= $\alpha_1, \alpha_2, \cdots, \alpha_n$.

Now U_{α_i} is open in X_{α_i} and $x_{\alpha_i} \in U_{\alpha_i}$ then there exists $B_{\alpha_i} \in B_{\alpha_i}$ such that

$x_{\alpha_i} \in B_{\alpha_i} \subset U_{\alpha_i}$

Define $Q_{\alpha\in J}$

B_α where $B_\alpha \in \mathrm{B}_\alpha$ for $\alpha = \alpha_1, \alpha_2, \cdots, \alpha_n$.

$B_\alpha = X_\alpha$ for α 6= $\alpha_1, \alpha_2, \cdots, \alpha_n$

Then clearly $\mathrm{Q}_{\alpha \in J}$

$B_\alpha \in 1'$ and

$x = (x_\alpha)_{\alpha \in J} \in B_\alpha \subset \mathrm{Q}_{\alpha \in J}$

$U_\alpha \subset V$ for all $x \in \mathrm{Q}_{\alpha \in J}$

X_α, there exists $\mathrm{Q}_{\alpha \in J}$

$B_\alpha \in 1'$ such

that $x \in \mathrm{Q}_{\alpha \in J}$

$B_\alpha \subset V$.

By 0.2.3, $1'$ is a basis for the product topology in QX_α.

Theorem 0.8.8. *Let A_α be a subspace of X_α, for each $\alpha \in J$. Then* QA_α *is a subspace of* QX_α *if both products are given the box topology, or if both products are given the product topology.*

Proof. By 0.8.7, QB_α is the basis for the subspace QA_α (since $A_\alpha \subset X_\alpha$).

Therefore, Q$A_\alpha \subset$ QX_α.

Theorem 0.8.9. *If each space X_α is a Hausdorff space, then* QX_α *is a Hausdorff space in both the box and product topologies.*

Proof. Write 0.8.6.

Since X_α is Hausdorff, then there are distinct neighborhoods in X_α.

Their product also containing disjoint neighborhoods.

Therefore, QX_α is Hausdorff. 2

Theorem 0.8.10. *Let $\{X_\alpha\}$ be an indexed family of spaces; let $A_\alpha \subset X_\alpha$ for each α. If* QX_α *is given either the product or the box topology, then* QA_α = QA_α*.*

Proof. Let $(x_\alpha) \in$ QA_α.

To show that $(x_\alpha) \in$ QA_α.

Let $U =$ QU_α be a basis elements for box or product topology that contains x.

Since $x = (x_\alpha) \in A_\alpha$, we can choose a point $y_\alpha \in U_\alpha \cap A_\alpha$.

Then $y = (y_\alpha) \in U$ and QA_α.

Since U is arbitrary, $(x_\alpha) \in$ QA_α.

Therefore, Q$A_\alpha \subseteq$ QA_α.

Conversely, suppose $(x_\alpha) \in$ QA_α.

To show that $(x_\alpha) \in$ QA_α.

Let $V_\beta \in X_\beta$ containing x_β.

By definition of product topology, since π^{-1}
$_{\beta}$ (V_β) is open in QX_α in either topology,

$x_\beta \in V_\beta \subset X_\beta$.

Then π^{-1}
$_{\beta}$ (V_β) is open in QX_α.

Since $A_\alpha \subset X_\alpha$, $y_\alpha \in QA_\alpha$.

Now $y_\beta \in V_\beta \cap A_\beta$

Then $x_\beta \in A_\beta$

$\Rightarrow (x_\beta) \in QA_\alpha$

$\Rightarrow QA_\alpha \subseteq QA_\alpha$

Therefore, $QA_\alpha = QA_\alpha$.

Theorem 0.8.11. *Let* $f : Q_{\alpha \in J}$
X_α *be given by the equation*
$f(a) = (f_\alpha(a))_{\alpha \in J}$,

where $f_\alpha : A \to X_\alpha$ *for each* α. *Let* QX_α *have the product topology. Then the fnction* f *is continuous if and only if each function* f_α *is continuous.*

Proof. Let $f : A \to Q_{\alpha \in J}$

X_α be given by $f(a) = (f_\alpha(a))_{\alpha \in J}$ where $f_\alpha : A \to X_\alpha$.

Let QX_α have the product topology.

Now let π_β be the projection of the product onto its β th factor.

ie, $\pi_\beta : Q_{\alpha \in J}$

$X_\alpha \to X_\beta$.

Therefore, the function π_β is continuous.

For, if U_β is open in X_β, the set π^{-1}
$_{\beta}$ (U_β) is a subbasis element for the product
topology on X_α.

Now suppose $f : A \to Q_{\alpha \in J}$

X_α is continuous.

Since π_β and f are continuous, the composite of these two maps, $\pi_\beta \circ f$ is continuous.

$\pi_\beta \circ f = f_\beta$ where $f_\beta : A \to X_\beta$ is continuous.

Therefore, f_β is continuous.

Conversely, suppose each function f_α is continuous.

To prove $f : A \to QX_\alpha$ is continuous.

π^{-1}
$_{\beta}$ (U_β) is a subbasis element for the product topology on QX_α, where U_β is
open in X_β.

$f^{-1}(\pi^{-1}$

β (U_β)) = ($\pi_\beta \circ f)^{-1}(U_\beta)$ = f^{-1}
β (U_β)

Since $f_\beta : A \to X_\beta$ is continuous, f^{-1}
β (U_β) is open in A.

$f^{-1}(\pi^{-1}$
β (U_β)) is open in A.

Therefore, f is continuous.

Unit - III

The Metric Topology

Defn:

A metric on a set X is a function $f: X \times X \to R$ having the following properties;

i). $d(x,y) \geq 0$ for all $x, y \in X$ and $d(x,y) = 0$ iff $x = y$.

ii). $d(x,y) = d(y,x)$ $\forall$ $x, y \in X$.

iii). (Triangle Inequality)

$d(x,y) + d(y,z) \geq d(x,z)$ $\forall$ $x, y, z \in X$.

Given a metric d on x, the number $d(x,y)$ is often called the distance b/w x and y in the metric d.

Defn:

Given $\varepsilon > 0$, the set $B_d(x, \varepsilon) = \{y \,/\, d(x,y) < \varepsilon\}$ is called the ε-ball centered at x.

Defn:

If d is a metric on the set X, then the collection of all ε-balls $B_d(x, \varepsilon)$ for $x \in X$ and $\varepsilon > 0$ is a basis for a topology on X, called the metric topology induced by d.

Defn:

A set U is open in the metric topology induced by d iff for each $y \in U$, there is a $\delta > 0$ such that $B_d(y, \delta) \subset U$.

Example: -1

Given a set X, define

$$d(x,y) = \begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{if } x = y. \end{cases}$$

Then d is a metric, the topology it induces is the discrete topology, the basis element x alone.

5.4 B(x,1) consists of the point

Note:-
$$(x, \varepsilon) = \begin{cases} \{x\}, & \text{for all } \varepsilon \le 1. \\ x, & \text{for all } \varepsilon > 1. \end{cases}$$

Example:- 2

The standard metric on the real numb[ers] $\mathbb{R}$ is defined by the equation,

$d(x, y) = |x - y|$.

Then d is a metric the topology it ind[uces] is the same is the order topology.

Each basis element (a, b) for the order topology is a basis element for the metric topology indeed $(a, b) = B(x, \varepsilon)$.

where $x = \frac{(a+b)}{2}$ and $\varepsilon = \frac{(b-a)}{2}$

and conversely,

each ε ball $B(x, \varepsilon)$ equals an op[en] interval the interval $(x - \varepsilon, x + \varepsilon)$.

Defn:- If X is a topological space, X is [said] to be metrizable if there exists a metric on the set X that induces topology of X.

A metric space is a metrizable space X together with a specific metric d that gi[ves] the topology of X.

Defn:- Let X be a metric space with metric [d]. A subset A of X is said to be bounded if there is some number M such that $d(a_1, a_2) \le M$. for every pair a_1, a_2 of A.

55 If A is bounded and nonempty the diameter of A is defined to be the number.

$\dim A = \sup \{d(a_1, a_2) / a_1, a_2 \in A\}$.

Theorem:- 20.1

Let X be a metric space with metric d. Define $\bar{d} : X \times X \to \mathbb{R}$ by the equation $\bar{d}(x, y) = \min \{d(x, y), 1\}$. Then $\bar{d}$ is a metric that induces the same topology as d. The metric $\bar{d}$ is called the standard bounded.

Proof:-

Claim: $\bar{d}$ is a metric.

i). $d(x, y) \geq 0$.

$\min \{d(x, y), 1\} \geq 0$.

i.e), $\bar{d}(x, y) \geq 0$.

Also, $\bar{d}(x, y) = 0$ iff $\min \{d(x, y), 1\} = 0$

iff $d(x, y) = 0$

iff $x = y$.

ii). $\bar{d}(x, y) = \min \{d(x, y), 1\}$

$= \min \{d(y, x), 1\}$

$= \bar{d}(y, x)$.

$\therefore \bar{d}(x, y) = \bar{d}(y, x)$.

iii). suppose $d(x, y) \geq 1$ (or) $d(y, z) \geq 1$.

Then $\min \{d(x, y), 1\} \geq 1$ (or)

$\min \{d(y, z), 1\} \geq 1$.

i.e), $\bar{d}(x, y) \geq 1$ (or) $\bar{d}(y, z) \geq 1$.

$\therefore \bar{d}(x, y) + \bar{d}(y, z) \geq 1$.

But $\bar{d}(x, z) = \min \{d(x, z), 1\} \leq 1$

$\leq \bar{d}(x,y) + \bar{d}(y,z)$.

$\therefore \bar{d}(x,z) \leq \bar{d}(x,y) + \bar{d}(y,z)$.

Now, suppose $d(x,y) < 1$ and $d(y,z) < 1$.

Then $\min\{d(x,y), 1\} = d(x,y)$ and

$\min\{d(y,z), 1\} = d(y,z)$

i.e., $\bar{d}(x,y) = d(x,y)$ and $\bar{d}(y,z) = d(y,z)$

$\therefore \bar{d}(x,z) \leq d(x,z)$

$\leq d(x,y) + d(y,z)$

$= \bar{d}(x,y) + \bar{d}(y,z)$

i.e., $\bar{d}(x,z) \leq \bar{d}(x,y) + \bar{d}(y,z)$

$\therefore$ The triangle inequality holds for $\bar{d}$.

Hence $\bar{d}$ is a metric on X.

To prove that $\bar{d}$ induces the same topology as d.

Let $\mathcal{T}_d$ and $\mathcal{T}_{\bar{d}}$ be the topologies induced by d and $\bar{d}$ respectively.

we have

$\mathcal{B}_d = \{B_d(x,\varepsilon) \mid x \in X, \varepsilon > 0\}$ and

$\mathcal{B}_{\bar{d}} = \{B_{\bar{d}}(x,\varepsilon) \mid x \in X, \varepsilon > 0\}$ are basis for $\mathcal{T}_d$ and $\mathcal{T}_{\bar{d}}$ respectively.

Let $x \in B_d(y, \delta)$

Then $d(y,x) < \delta$.

But $\bar{d}(y,x) = \min\{d(y,x), 1\}$

$= d(y,x)$

Hence, $\bar{d}(y,x) < \delta$.

$\therefore x \in B_{\bar{d}}(y,\delta)$

Thus, we have, $x \in B_d(y,\delta)$

$\subset B_{\bar{d}}(y,\delta)$.

$\therefore \mathcal{T}_{\bar{d}} \subset \mathcal{T}_d$ —— (1)　　57

Now, let $B_d(x, \varepsilon) \in \mathcal{B}_d$ and $y \in B_d(x,\varepsilon)$

Choose $\delta < 1$ such that

$B_d(y, \delta) \subset B_d(x, \varepsilon)$ —— (2).

clearly, $B_d(y, \delta) \in \mathcal{B}_{\bar{d}}$ and $y \in B_{\bar{d}}(y, \delta)$.

To prove: $B_{\bar{d}}(y, \delta) \subset B_d(x, \varepsilon)$.

let $z \in B_{\bar{d}}(y, \delta)$.

then, $\bar{d}(y, z) < \delta < 1$.

$\bar{d}(y, z) = \min\{d(y, z), 1\}$

$= d(y, z)$.

$\Rightarrow d(y, z) < \delta$.

$\Rightarrow z \in B_d(y, \delta) \subset B_d(x, \varepsilon)$.

$\therefore B_{\bar{d}}(y, \delta) \subset B_d(x, \varepsilon)$

Hence $\mathcal{T}_d \subset \mathcal{T}_{\bar{d}}$ —— (3)

From (1) and (3), we get

$\mathcal{T}_d = \mathcal{T}_{\bar{d}}$.

Definition:

Given $x = (x_1, x_2, \ldots, x_n)$ in R^n,

we define the norm of x by the equation

$\|x\| = (x_1^2 + \ldots + x_n^2)^{1/2}$;

and we define the euclidean metric d on R^n

by the equation

$d(x, y) = \|x - y\| = \left[(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2\right]^{1/2}$

We define the square metric ρ by the

equation, $\rho(x, y) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}$

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Note: On the real line $R = R'$; these two metric coincide with the standard metric for R. In the plane R^2, the basis elements under ~~circles~~ can be picture as circular regions while the basis elements under β can be pictured as square regions.

Lemma:- 20.2

Let d and d' be two metrics on the X; let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies they induce respectively. Then $\mathcal{T}'$ is finer than $\mathcal{T}$ iff for each x in X and each $\varepsilon > 0$, there exists a $\delta > 0$ such that $B_{d'}(x, \delta) \subset B_d(x, \varepsilon)$

Proof:

Assume that $\mathcal{T}'$ is finer than $\mathcal{T}$. Consider the basis element $B_d(x, \varepsilon)$ for $\mathcal{T}$. By lemma, there is a basis element B' for the topology $\mathcal{T}'$ such that $x \in B' \subset B_d(x, \varepsilon)$ with in B' we can find a ball $B_{d'}(x, \delta)$ centred at x.

for, let $B' = B_{d'}(y, \delta_1)$ for some $y \in X$ and $\delta_1 > 0$.

Now, $x \in B_{d'}(y, \delta_1)$.

Take $\delta = \delta_1 - d'(x, y)$.

Claim: $B_{d'}(x, \delta) \subset B'$.

$z \in B_{d'}(x, \delta) \Rightarrow d'(x, z) = \delta$.

$d'(z, y) \leq d'(z, x) + d'(x, y)$

$< \delta + \delta_1 - \delta$

$= \delta_1$.

$z \in B_{d'}(y, \delta_1) = B'$.

$\Rightarrow x \in B'$.

Thus $B_{d'}(x, \delta) \subset B'$.

$\Rightarrow x \in B_{d'}(x, \delta) \subset B' \subset B_d(x, \varepsilon)$.

Then there is a $\delta > 0$ such that

$x \in B_{d'}(x, \delta) \subset B_d(x, \varepsilon)$.

Conversely,

Assume that for each $x \in X$ and each $\varepsilon > 0$, there is a $\delta > 0$ such that

$B_{d'}(x, \delta) \subset B_d(x, \varepsilon)$.

<u>To prove:</u> $\mathcal{T} \subset \mathcal{T}'$.

Let $B = B_d(x, \varepsilon)$ for some $x \in X$ and $\varepsilon > 0$.

Let $y \in B_d(x, \varepsilon)$.

Let $\varepsilon = \varepsilon - d(x, y)$

Consider $B_d(y, \varepsilon_1)$.

<u>Claim:</u> - $B_d(y, \varepsilon_1) \subset B$.

Let $z \in B_d(y, \varepsilon_1)$

Then $d(y, z) < \varepsilon_1$.

$d(x, z) \leq d(x, y) + d(y, z)$

$< \varepsilon - \varepsilon_1 + \varepsilon_1$

$= \varepsilon$.

$\Rightarrow z \in B_d(x, \varepsilon) = B$.

Thus $B_d(y, \varepsilon_1) \subset B$.

By hypotheses, There is a $\delta > 0$ such that $B_{d'}(y, \delta) \subset B_d(x, \varepsilon)$.

Let $B' = B_{d'}(y, \delta)$

Then $y \in B' = B_{d'}(y, \delta)$

$\subset B_d(x, \varepsilon)$

$= B$,

i.e) $y \in B' \subset B$.

$\therefore$ By lemma, $\mathcal{T}'$ is finer than $\mathcal{T}$.

Theorem:-
The topologies on $\mathbb{R}^n$ induced by the euclidean metric d and square metric ρ are the same as the product topology on $\mathbb{R}^n$.

Proof:-
Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be two points on $\mathbb{R}^n$.

claim:- $\rho(x,y) \leq d(x,y) = \sqrt{n}\, \rho(x,y)$

Now, $|x_i - y_i|^2 \leq (|x_1 - y_1|)^2 + (x_2 - y_2)^2 + \cdots + |x_n - y_n|^2$

$\therefore |x_i - y_i| \leq \sqrt{(|x_1 - y_1|)^2 + |x_2 - y_2|^2 + \cdots + |x_n - y_n|^2}$

i.e), $|x_i - y_i| \leq d(x,y)$ for $i = 1, 2, \ldots n$

$\therefore \max\{|x_1 - y_1|, |x_2 - y_2|, \ldots |x_n - y_n|\} \leq d(x,y)$

i.e), $\rho(x,y) \leq d(x,y)$ —— ①

Suppose $\rho(x,y) = \max\{|x_1 - y_1|, \ldots |x_n - y_n|\}$
$= |x_i - y_i|$ (say)

Then $|x_j - y_j| \leq |x_i - y_i|$, $\forall\ 1 \leq j \leq n$.

$\Rightarrow |x_j - y_j|^2 \leq |x_i - y_i|^2$

$\Rightarrow \sum_{j=1}^{n} |x_j - y_j|^2 \leq n\, |x_i - y_i|^2$.

$\Rightarrow \sqrt{\sum_{j=1}^{n} |x_j - y_j|^2} \leq \sqrt{n}\, |x_i - y_i|$

$\Rightarrow d(x,y) \leq \sqrt{n}\, \rho(x,y)$ —— ②

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From ① and ②, we get

$\rho(x,y) \le d(x,y) \le \sqrt{n}\,\rho(x,y)$ —— ③.

Let $\mathcal{T}_d$, $\mathcal{T}_\rho$ be the metric topologies induced by d and ρ respectively.

Let $\mathcal{T}$ be the product topology.

T.P:- $\mathcal{T}_d = \mathcal{T}_\rho = \mathcal{T}$.

First, we prove :- $\mathcal{T}_d = \mathcal{T}_\rho$.

Now $d(x,y) < \varepsilon \Rightarrow \rho(x,y) < \varepsilon$ (by ③).

$B_d(x,\varepsilon) \subset B_\rho(x,\varepsilon)$, for all x and ε.

$\therefore \mathcal{T}_\rho \subset \mathcal{T}_d$ —— ④.

Now, $\rho(x,y) < \frac{\varepsilon}{\sqrt{n}} \Rightarrow d(x,y) < \varepsilon$ (by ③)

$\therefore B_\rho(x, \varepsilon/\sqrt{n}) \subset B_d(x,\varepsilon)$ for all x and ε.

$\therefore \mathcal{T}_d \subset \mathcal{T}_\rho$ —— ⑤.

From ④ and ⑤, we get,

$\mathcal{T}_d = \mathcal{T}_\rho$.

Now, we show that $\mathcal{T}_\rho = \mathcal{T}$.

Let $B = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n)$

be a basis element for $\mathcal{T}$.

Let $x \in B$, where $x = (x_1, x_2, \ldots x_n)$.

For each i, there is an ε_i such that

$(x_i - \varepsilon_i, x_i + \varepsilon_i) \subset (a_i, b_i)$.

choose $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\}$

Then, $(x_1 - \varepsilon, x_1 + \varepsilon) \times (x_2 - \varepsilon, x_2 + \varepsilon) \times \cdots \times (x_n - \varepsilon, x_n + \varepsilon)$.

$\subset (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n)$

$= B$ —— ⑥

Now, let $y \in B_\rho(x, \varepsilon)$.

then $\rho(x, y) < \varepsilon$.

$\Rightarrow |x_i - y_i| < \varepsilon, \forall i$

$\Rightarrow y_i \in (x_i - \varepsilon, x_i + \varepsilon) \subset (a_i, ...$

$\Rightarrow y \in B$ (by ⑥)

$\therefore B_\rho(x, \varepsilon) \subset B$

Thus $\mathcal{T} \subset \mathcal{T}_\rho$.

Now, let $B_\rho(x, \varepsilon)$ be a basis element for the ρ-topology.

Let $y \in B_\rho(x, \varepsilon)$

Then $\rho(x, y) < \varepsilon$.

$\Rightarrow |x_i - y_i| < \varepsilon, \forall i$

$\Rightarrow y_i \in (x_i - \varepsilon, x_i + \varepsilon)$

$\Rightarrow y_i \in (x_1 - \varepsilon, x_1 + \varepsilon) \times \dots \times (x_n - \varepsilon, x_n + \varepsilon)$

$\Rightarrow y \in B$.

$\therefore B_\rho(x, \varepsilon) \subset B$.

then $\mathcal{T} \subset \mathcal{T}_\rho$ —— ⑧

From ⑦ and ⑧, we get

$\mathcal{T} = \mathcal{T}_\rho'$.

Thus $\mathcal{T} = \mathcal{T}_d = \mathcal{T}_\rho$.

Defn: Given an index set J, and given points $(x_\alpha)_{\alpha \in J}$ and $(y_\alpha)_{\alpha \in J}$ of $\mathbb{R}^J$, define a metric $\bar{\rho}$ on $\mathbb{R}^J$ by the equation, $\bar{\rho}(x, y) = \sup \{ \bar{d}(x_\alpha, y_\alpha) / \alpha \in J \}$

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where $\bar{d}$ is the standard bounded metric on $\mathbb{R}$.

The metric $\bar{\rho}$ is called the uniform metric on $\mathbb{R}^J$, and the topology it induces is called the uniform topology.

Theorem: The uniform topology on $\mathbb{R}^J$ is finer than the product topology and coarser than the box topology; these three topologies are all different if J is infinite.

$\mathcal{T}_p \subset \mathcal{T}_{\bar{\rho}} \subset \mathcal{T}_b$

Proof: Let $x = (x_\alpha)_{\alpha \in J} \in \mathbb{R}^J$.

Take any basis element for the product topology $\prod_{\alpha \in J} U_\alpha$ containing x.

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be indices for which $U_\alpha \neq \mathbb{R}$.

Each U_{α_i} is open in $\mathbb{R}$.

For each i, choose ε_i so that

$B_{\bar{d}}(x_{\alpha_i}, \varepsilon_i) \subset U_{\alpha_i}$.

Let $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\}$

Claim: $B_{\bar{\rho}}(x, \varepsilon) \subset \prod_{\alpha \in J} U_\alpha$.

Let $y = (y_\alpha)_{\alpha \in J} \in B_{\bar{\rho}}(x, \varepsilon)$.

$\Rightarrow \bar{\rho}(x, y) < \varepsilon$

$\Rightarrow \sup\{\bar{d}(x_\alpha, y_\alpha) / \alpha \in J\} < \varepsilon$.

i.e), $\bar{d}(x_\alpha, y_\alpha) < \varepsilon_i < \varepsilon$, $i = 1, 2, \ldots n$

$\Rightarrow y_{\alpha_i} \in B_{\bar{d}}(x_{\alpha_i}, \varepsilon_i)$

$\Rightarrow B_{\bar{d}}(x_\alpha, \varepsilon_i) \in U_{\alpha_i}$

$\Rightarrow y_{\alpha_i} \in U_{\alpha_i}$

$\Rightarrow y_\alpha \in U_\alpha$

$\Rightarrow y \in \prod_{\alpha \in J} U_\alpha$.

$\therefore B_{\bar{\rho}}(x, \varepsilon) \subset \prod_{\alpha \in J} U_\alpha$.

$\therefore$ the uniform topology is finer than product topology.

On the other hand, let B be the $\bar{\rho}$ metric. ε-ball centred at x is the

Then the box neighbourhood

$U = \prod (x_\alpha - \varepsilon/2, x_\alpha + \frac{\varepsilon}{2})$ of x

is contained in B. then $\bar{d}(x_\alpha, y_\alpha) < \frac{\varepsilon}{2}$.

For, if $y \in U$, then

for all α, so that $\bar{\rho}(x, y) \le \frac{\varepsilon}{2}$.

Therefore, the uniform topology is coarser than the box topology.

Let J be infinite.

$x \in B_{\bar{\rho}}(0, 1/2) \Leftrightarrow \bar{d}(0, x) < 1/2$

$\Leftrightarrow \bar{d}(0, x_n) < 1/2$

$\Leftrightarrow |x_n| < 1/2$

$\Leftrightarrow x_n \in (-1/2, 1/2)$

$\therefore B_{\bar{\rho}}(0, 1/2) = \prod U_n, \quad U_n = (-1/2, 1/2)$.

This is a basis open set in the uniform topology but not open in product topology.

Take a basis element for the product topology say $\prod_{\alpha \in J} U_\alpha$, where $U_\alpha = \mathbb{R}$, $\forall \alpha \neq \alpha_1, \ldots, \alpha_n$.

Consider, the basic open set $B_{\bar{\rho}}(0, 1)$.

Now $x = (x_\alpha)_{\alpha \in J} \in B_{\bar{\rho}}(0,1)$.

$\iff \bar{\rho}(0,x) < 1$.

$\iff \bar{d}(0,x_n) < 1$

$\iff |x_n| < 1$

$\iff x_n \in (-1,1)$.

$\iff$

Thus $B_{\bar{\rho}}(0,1) = \prod U_n$, $U_n = (-1,1)$.

<u>claim:</u> $\prod_{n \in \mathbb{Z}_+} U_n$, $U_n = (-1,1)$, $\forall n$ is not open in the product topology.

If $\prod U_n$ is open in the product topology, then $0 \in \prod_{n \in \mathbb{Z}_+} V_n \subset \prod U_n$,

where $V_n = \mathbb{R}$ except for finitely n.

Let $n_0 \in \mathbb{Z}_+$, be such that $V_{n_0} = \mathbb{R}$.

Now, $V_{n_0} \subset U_{n_0} = (-1,1)$

i.e), $\mathbb{R} \subseteq (-1,1)$.

which is a contradiction.

$\therefore \prod U_n$ is not open in the product topology.

i.e), $B_{\bar{\rho}}(0,1)$ is not open in the product topology.

$\therefore$ The uniform topology on $\mathbb{R}^J$ is strictly finer than the product topology.

Theorem: Let $\bar{d}(a,b) = \min\{|a-b|, 1\}$ be the standard bounded metric on $\mathbb{R}$. If x and y are two points of $\mathbb{R}^\omega$, define

$$D(x,y) = \sup\left\{\frac{\bar{d}(x_i, y_i)}{i}\right\}$$

Then D is a metric that induces the product topology on $\mathbb{R}^\omega$.

Proof:

To prove D is a metric

i). Now $\bar{d}(x_i, y_i) \geq 0$ for all i.

$$\therefore \sup\left\{\frac{\bar{d}(x_i, y_i)}{i}\right\} \geq 0$$

i.e., $D(x,y) \geq 0$.

Also $D(x,y) = 0 \Rightarrow \sup\left\{\frac{\bar{d}(x_i, y_i)}{i}\right\} = 0$

$\Leftrightarrow \bar{d}(x_i, y_i) = 0, \forall i$

$\Leftrightarrow x_i = y_i, \forall i$

$\Leftrightarrow x = y$.

ii). $D(x,y) = \sup\left\{\frac{\bar{d}(x_i, y_i)}{i}\right\}$

$= \sup\left\{\frac{\bar{d}(y_i, x_i)}{i}\right\}$

$= D(y,x)$

iii). Let $x, y, z \in \mathbb{R}^\omega$.

Now $\frac{\bar{d}(x_i, z_i)}{i} \leq \frac{\bar{d}(x_i, y_i)}{i} + \frac{\bar{d}(y_i, z_i)}{i}$

$= D(x,y) + D(y,z)$ for all i

$\Rightarrow \sup\left\{\frac{\bar{d}(x_i, y_i)}{i}\right\} \leq D(x,y) + D(y,z)$

$\Rightarrow D(x,z) \leq D(x,y) + D(y,z)$.

∴ D is a metric.

claim: D induces the product topology.

Let U be an open set in the metric, topology and $x \in U$. we find an open set V in the product topology such that $x \in V \subset U$.

Choose an ε-ball $B_D(x, \varepsilon)$ lying in U. Then choose N large enough that $\frac{1}{N} < \varepsilon$. Finally, let V be the bases element for the product topology.

~~$V = (x_1 - \varepsilon)$~~

$V = (x_1 - \varepsilon, x_1 + \varepsilon) \times \ldots \times (x_N - \varepsilon, x_N + \varepsilon) \times \mathbb{R} \times \mathbb{R} \ldots$

we prove that $V \subset B_d(x, \varepsilon)$.

Given y in $\mathbb{R}^{\omega}$.

$$\frac{d(x_i, y_i)}{i} = \frac{1}{N} \text{ for all } i \geq n.$$

therefore, $D(x, y) \leq \max \left\{ \frac{d(x_1, y_1)}{1}, \ldots \frac{d(x_N, y_N)}{N}, \frac{1}{N} \right\}$

If y is in V, $D(x, y) < \varepsilon$ so that

$V \subset B_D(x, \varepsilon)$

∴ $x \in V \subset U$.

Conversely, consider a bases element $U = \prod_{i \in Z} U_i$, for the product topology, where U_i is open in $\mathbb{R}$ for $i \neq \alpha_1, \alpha_2, \ldots \alpha_n$ and $U_i = \mathbb{R}$ for all others indices i.

Given $x \in V$, we find an open set $x \in V \subset U$.

Choose an interval $(x_i - \varepsilon_i, x_i + \varepsilon_i)$ in $\mathbb{R}$ centred about x_i and laying in U_i for $i = 1, \ldots, n$; choose each $\varepsilon_i \leq 1$. Then define,

$$\varepsilon = \min\left\{ \frac{\varepsilon_i}{i}, \; i = 1, \ldots, n \right\}$$

we prove that $x \in B_D(x, \varepsilon) \subset U$.

Let y be a point of $B_D(x, \varepsilon)$

Then for all i,

$$\frac{\bar{d}(x_i, y_i)}{i} \leq D(x, y) < \varepsilon.$$

Now, if $i = 1, \ldots, n$ then $\varepsilon < \frac{\varepsilon_i}{i}$

$\bar{d}(x_i, y_i) < \varepsilon_i \leq 1.$

$\Rightarrow |x_i - y_i| < \varepsilon_i < \varepsilon_i$

$\Rightarrow y \in \prod U_i$

$\therefore x \in V \subset U.$

Hence D induces the product topology on $\mathbb{R}^\omega$.

The metric topology (continued)

Theorem:-

Let $f: X \to Y$. Let X and Y be metriz with metrices d_X and d_Y respectively. Then continuity of f is equivalent to the requirement that given $x \in X$ and given $\varepsilon > 0$, $\exists$ $\delta > 0$ such that $d_X(x, y) < \delta \Rightarrow$

$d_Y(f(x), f(y)) < \varepsilon$.

proof:-

Suppose that f is continuous.

Let $x \in X$ and $\varepsilon > 0$ be given.

Consider the set $f^{-1}(B_{d_Y}(f(x), \varepsilon))$.

$\Rightarrow B_{d_Y}(f(x), \varepsilon)$ is open in Y.

$\therefore f^{-1}(B_{d_Y}(f(x), \varepsilon))$ is open in X.

and $x \in X$.

$\therefore \exists$ a $\delta > 0$ such that

$B_{d_X}(x, \delta) \subset f^{-1}(B_{d_Y}(f(x), \varepsilon))$

$\Rightarrow f(B_{d_X}(x, \delta)) \subset B_{d_Y}(f(x), \varepsilon)$

Thus, if $d_X(x, y) < \delta$, then $y \in B_{d_X}(x, \delta)$

$\therefore f(y) \in f(B_{d_X}(x, \delta))$

$\subset B_{d_Y}(f(x), \varepsilon)$

i.e., $d_Y(f(x), f(y)) < \varepsilon$.

Thus we have found $\delta > 0$ such that

$d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$.

Conversely,

Suppose that for any given $x \in X$ and $\delta > 0$, $\exists$ a $\delta > 0$ such that

$d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$.

T.P:- f is continuous.

Let V be any open set in Y.

we show that $f^{-1}(V)$ is open in X.

Let $x \in f^{-1}(V)$.

Then $f(x) \in V$.

$\therefore \exists$ a $\varepsilon > 0$ such that $B_{d_Y}(f(x), \varepsilon) \subset V$.

By hypothesis, $\exists$ a $\delta > 0$ such that

$d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$.

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Consider the open ball $B_{d_X}(x, \delta)$

claim:- $B_{d_X}(x, \delta) \subset f^{-1}(V)$.

Let $y \in B_{d_X}(x, \delta)$

$\Rightarrow d_X(x, y) < \delta$

$\Rightarrow d_Y(f(x), f(y)) < \varepsilon$.

$\Rightarrow f(y) \in B_{d_Y}(f(x), \varepsilon) \subset V$.

$\Rightarrow f(y) \in V$.

$\Rightarrow y \in f^{-1}(V)$.

Thus $B_{d_X}(x, \delta) \subset f^{-1}(V)$

Hence $f^{-1}(V)$ is open in X.

$\therefore$ f is continuous.

Lemma:- (The sequence Lemma)

Let X be a topological space; le[t] $A \subset X$. If there exists a sequence of p[oints] of A convergins to x, then $x \in \bar{A}$, the Converse holds if X is metrizable.

Proof:- Suppose that $(x_n) \to x$, where $x_n \in A$

To show that $x \in \bar{A}$.

Let U be any neighbourhood of x.

Since $x_n \to x$, there exists N such tha[t] $x_n \in U$, $\forall n \geq N$.

i.e), $x_N \in U$.

$\therefore U \cap A \neq \phi$.

Thus every neighbourhood of x intersect[s] A.

$\therefore x \in \bar{A}$.

For Converse, let X be metrizable.

Let $x \in \bar{A}$.

For each positive integer n, Consider the neighbourhood $B_d(x, \frac{1}{n})$ of radius $\frac{1}{n}$ of x

Now $B_d(x, \frac{1}{n}) \cap A \neq \phi$.

Choose $x_n \in B_d(x, \frac{1}{n}) \cap A$.

<u>Claim</u>:- $(x_n) \to x$.

Let V be any neighbourhood of x.

There exists $\varepsilon > 0$, such that

$B_d(x, \varepsilon) \subset V$.

Choose N such that $\frac{1}{N} < \varepsilon$.

Then $B_d(x, 1/n) \subset B_d(x, \frac{1}{N}) \subset B_d(x, \varepsilon)$.

$\therefore B_d(x, \frac{1}{n}) \subset B_d(x, \varepsilon)$, $\forall n \geq N$.

i.e), $x_n \in B_d(x, \varepsilon)$, $\forall n \geq N$.

i.e), $d(x_n, x) < \varepsilon$, $\forall n \geq N$.

$\therefore x_n \to x$.

<u>Theorem</u>:-

Let $f: X \to Y$. If the function f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if X is metrizable.

<u>Proof</u>:- Assume that f is continuous.

Let (x_n) be sequence of points of X.

Converges to x.

To show that :- $f(x_n) \to f(x)$.

Let V be any neighbourhood of $f(x)$.

Then $f^{-1}(V)$ is a neighbourhood of x.

Now, $x_n \to x$ and $f^{-1}(V)$ is a neighbourhood of x.

$\therefore$ There exists N such that

$x_n \in f^{-1}(V)$, $\forall n \geq N$.

i.e), $f(x_n) \in V$, $\forall n \geq N$.

$\therefore f(x_n) \to f(x)$.

Conversely,

Assume that for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n) \to f(x)$.

To prove:- f is continuous.

we will show that $f(\overline{A}) \subset \overline{f(A)}$, where A is a subset of X.

Let $x \in \overline{A}$.

T.P:- $f(x) \in \overline{f(A)}$.

Now, X is metrizable.

By sequence lemma, there exists a sequence x_n of points of A conversing to x.

By assumption, $f(x_n) \to f(x)$.

Since, $f(x_n) \in f(A)$, by the sequence lemma, we have $f(x) \in \overline{f(A)}$.

Hence, $f(\overline{A}) \subset \overline{f(A)}$.

$\therefore f$ is continuous.

Definition:- (**Countable basis**)

A space X is said to have a "Countable basis" at the point x if there is a countable collection $\{\ldots\}$

$\{U_n\}_{n \in Z_+}$ of neighbourhoods of x contains at least one the sets U_n.

A space X that has a countable basis at each of its point is said to satisfy the first countability axiom.

Definition:- (Converges uniformly) (or) (uniformly converges)

Let $f_n : X \to Y$ be a sequence functions from set X to the metric spaces Y. Let d be the metric for Y. We say that the sequence (f_n) "converges uniformly" to the function $f : X \to Y$ if given $\varepsilon > 0$, there exists an integer N such that

$$d(f_n(x), f(x)) < \varepsilon.$$

(for all $n \geq N$ and for all x in X.

Theorem:- (Uniform limit Theorem)

Let $f_n : X \to Y$ be a sequences of continuous functions from the topological space X to the metric space Y. If (f_n) converges uniformly to f, then f is continuous.

Proof:-

To prove:- f is continuous.

It is enough to prove that, for each $x_0 \in Y$ and each neighbourhood V of $f(x)$, there is a neighbourhood U of x_0 such that $f(U) \subset V$.

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Let $x_0 \in X$.

Let V be an open neighbourhood of $f(x_0)$. First, choose $\varepsilon > 0$ so that $B(f(x_0), \varepsilon) \subset V$. Then using uniform convergence, choose N so that for all $n \geq N$ and all $x \in X$,

$$d(f_n(x), f(x)) < \varepsilon/3 \quad \text{--- (1)}.$$

Finally, using continuity of f_N choose a neighbourhood U of x_0, such that

$$d(f_N(x), f_N(x_0)) < \varepsilon/3 \quad \text{--- (2)}$$

By the choice of N, we have

$$d(f_N(x_0), f(x_0)) < \varepsilon/3, \ \forall n \geq N, x_0 \quad \text{--- (3)}$$

By the triangular inequality, we have

$$d(f(x), f(x_0)) \leq d(f(x), f_N(x)) + d(f_N(x), f_N(x_0)) + \lambda(f_N(x_0), f(x_0))$$

$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

(by (1), (2), (3)).

i.e), $d(f(x), f(x_0)) < \varepsilon$.

$\Rightarrow f(U) \subset V$.

Hence, f is continuous.

Ex:- 2

$\mathbb{R}^\omega$ in the box topology is not metrizable

Sol:-

Let A be the subset of $\mathbb{R}^\omega$ consisting of those points all of whose coordinates are positive.

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$A = \{(x_1, x_2, \dots) \mid x_n > 0$ for all $x \in \mathbb{Z}_+\}$

Let $\overline{0} = (0, 0, \dots)$ be the origin in $\mathbb{R}^\omega$.

To prove :- $\overline{0} \in \overline{A}$, in the box topology.

i.e), To prove every neighbourhood of $\overline{0}$. intersects A.

Let $B = (a_1, b_1) \times (a_2, b_2) \times \dots$ be any basis element ~~Containg~~ Containing $\overline{0}$.

Then B intersects A, because the point $\left(\frac{b_1}{2}, \frac{b_2}{2}, \dots\right) \in B \cap A$.

Now we prove that there is no sequence of points of A converging to $\overline{0}$.

Let (a_n) be a sequence of points of A, where $a_n = (x_{1n}, x_{2n}, \dots, x_{in}, \dots)$

~~Every neighbourhood x_{in}~~.

Every coordinate x_{in} is positive, so we can construct a basis element B' for the box topology on $\mathbb{R}$ by setting

$B' = (-x_{11}, x_{11}) \times (-x_{22}, x_{22}) \times \dots$

Then B' continuous the origin $\overline{0}$, but it contains no member of the sequence (a_n).

$\Rightarrow a_n \notin B'$, because $x_n \notin (-x_{nn}, x_{nn})$.

Hence (a_n) can not converse to $\overline{0}$ in the box topology.

Hence $\mathbb{R}^\omega$ is not metrisable in the box topology.

Ex:- 2

An uncountable product of $\mathbb{R}$ with itself is not metrizable.

proof:- Let J be an uncountable index set. We show that $\mathbb{R}^J$ does not satisfy the sequence lemma (in the product topology) i.e), there is no sequence of points of A converging to x, where $A \subset \mathbb{R}^J$ and $x \in \bar{A}$.

Let $A = \{(x_\alpha) / x_\alpha = 1, \forall$ but finitely many values of $\alpha\}$.

Let $\bar{0} = (0, 0, \dots)$ be the origin in $\mathbb{R}^J$.

To prove:- $0 \in \bar{A}$.

Let πU_α be a base element containing $\bar{0}$.

Then $U_\alpha \neq R$ for finitely many values of α, say $\alpha = \alpha_1, \dots, \alpha_n$.

Let ~~(a) (a)~~ (x_α) be the points of A defined by $x_\alpha = 0$ for $\alpha = \alpha_1, \dots, \alpha_n$ and $x_\alpha = 1$ for all other values of α. Then $(x_\alpha) \in A \cap \pi U_\alpha$.

Hence $\bar{0} \in \bar{A}$.

Now to prove there is no sequence of points of A ~~conver~~ convergencing of ~~sequence~~ to $\bar{0}$.

Let (a_n) be a sequence of points of A. Given n, consider the subset $J_n \subset J$.

UNIT-IV

0.10 Connected spaces

Definition 0.10.1. *Let X be a topological space. A separation of X is a pair (u, v) of disjoint non empty open subsets of X whose union is X.*
Definition 0.10.2. *The space X is said to be connected if there dose not exists a separation of X.*
Remark 0.10.3. *If X is connected, then any space homomorphic to X is connected.*
Theorem 0.10.4. *A space X is connected iff the only subsets of X that are both open and closed are the empty set and X itself.*
Proof. First assume X is connected.
Claim : The only subsets of X that are both open and closed are the empty set and X itself.
For, suppose A is a nonempty proper subset of X. That is both open and closed in X.
We have $X - A$ is nonempty. If we take A is closed in X. Then $X - A$ is open.
Therefore we have two nonempty disjoint open sets A and $X - A$ such that their union is X.
That is A and $X - A$ forms a separation of X.
$\Rightarrow X$ is not conncted.
This contradication asserts our claim.
Conversely, assume the only subsets of X that are both open and closed are empty and X itself.
Claim : X is connected.
For, if X is not connected, there is a separation of X.
Let U and V forms the separation. Therefore U is nonempty.
U is open $\Rightarrow X - U$ is closed in X.
$\Rightarrow V$ is closed in X.
Also, V is open $\Rightarrow X - V$ is closed in X.
$\Rightarrow U$ is closed in X.
Thus we have U is a proper subset of X. That is both open and closed.
This is a contradication.
Therefore X is connected.
Lemma 0.10.5. *If Y is a subspace of X, a separation of Y is a pair of disjoint nonempty sets A and B whose union is Y , neither of which contains a limit point of the other. The space Y connected if there exists no separation of Y .*
Proof. Let Y be a subspace of X.
To prove separation of Y iff A and B are two nonempty disjoint sets such that

$A \cup B = Y$, $A \cap B = A \cap B = \varnothing$.

First assume that there exists a separation of Y. Then there exists disjoint nonempty open subsets A and B such that $A \cup B = Y$.

It is enough to prove $A \cap B = \varnothing$ and $A \cap B = \varnothing$.

Then A is both open and closed in Y.

The closure of A in Y is $A \cap Y$ where A denote the closure of A in Y.

Since Anis closed in X. $A = A \cap Y$ where A is the closure of A in X. To say the same thing $A \cap B = \varnothing$. Since A is the union of A and its limit points, B contains no limit points of A.

Similarly, we can show that A conatins no limit points of B.

Conversely, assume A and B are two nonempty disjoint sets such that $A \cap B = Y$, $A \cap B = A \cap B = \varnothing$.

Claim :$A \cap Y = A$.

We have A is contained A and $A \subset Y$.

That is $A \subset A$ and $A \subset Y$.

Therefore $A \subset A \subset Y$————————-(1)

Now, let $x \in A \subset Y$. Then $x \in A$ and $x \in Y$.

Therefore, $x \notin B$ and $x \in Y$.

$\Rightarrow x \in A$ (since $Y = A \cup B$).

Therefore, $A \cap Y \subset A$ ————————-(2).

From (1) and (2) we get, $A = A \cap Y$.

Similarlly, we can prove $B \cap Y = B$.

Now, A is closed in X.

$\Rightarrow A \cap Y$ is closed in Y.

$\Rightarrow A$ is closed in Y.

Similarlly, B is closed in Y.

Now, $B = Y - A$ is open in Y.

Therefore, B is open in Y.

Also $A = Y - B$.

Therefore, A is open in Y.

Thus A and B are two nonempty disjoint open sets in Y with $Y = A \cup B$.

Thus there exists a separation of Y.

Lemma 0.10.6. *If the sets C and D form a separation of X and if Y is connected subspace of X, then Y lies entirely with in either C or D.*

Proof. Let sets C and D form a separation of X.

Therefore, $X = C \cup D$ where C and D are nonempty disjoint open sets in X.
Let Y be a connceted subspace of X.
To prove Y lies entirely with in either C or D.

Since C and D are open in X, the sets $C \cap Y$ and $D \cap Y$ are open in Y.

Also, $Y = Y \cap X$

$= Y \cap (C \cup D)$

$= (Y \cap C) \cup (Y \cap D)$.

Now, $(Y \cap C) \cap (Y \cap D) = Y \cap (C \cap D)$

$= Y \cap \emptyset$

$= \emptyset$

Therefore, these two sets are disjoint and their union is Y.

If $C \cap Y$ and $D \cap Y$ are both nonempty.

Then they would constitute a separation of Y. Since Y is connceted, the only posibility is $Y \cap C = \emptyset$ or $Y \cap D = \emptyset$. Therefore, $Y \subset C$ or $Y \subset D$. Thatis, Y is entirely either in C or in D.

Example 0.10.7. *Let X denote a two points space in the indiscrete topology. Obviously there is no separation of X, so X is connected.*

Example 0.10.8. *Let Y denote the subspace* $[-1, 0) \cup (0, 1]$ *of the real line R each of the sets* $[-1, 0)$ *and* $(0, 1]$ *is nonempty and open in Y. They form a separation of Y.*

Example 0.10.9. *Let X be the subspace* $[-1, 1]$ *of the real line. The sets* $[-1, 0)$ *and* $(0, 1]$ *are disjoint and nonempty, but they does not form the separation of X. Because the first set is not open in X.*

Example 0.10.10. *The rationals Q are not connected.*

Lemma 0.10.11. *The union of a collection of connected subspaces of X that have a point in common is connected.*

Proof. Let $\{A_\alpha\}_{\alpha \in J}$ be a collection of connected subspaces of X that have a common point. Let $p \in A_\alpha$ for each α be the common point. To prove SA_α is connected. Let $Y = SA_\alpha$.

Suppose Y is not connected. Then there is a separation of Y. That is there exixt C and D are two nonempty disjoint open sets in Y such that $C \cup D = Y$.

We have $p \in Y$, therefore $p \in C$ or $p \in D$.

For, definteness let $p \in C$

Therefore, we have $p \in A_\alpha$

$\Rightarrow A_\alpha \subset C$ for each α

$\Rightarrow SA_\alpha \subset C$

That is $Y \subset C$

$\Rightarrow D$ is empty.

This is a contradication to D is nonempty. Therefore, Y is connceted. Thae is SA_α is connected.

Theorem 0.10.12. *Let A be a connected subspace of X and if* $A \subset B \subset \bar{A}$*. Then B is also connected.*

Proof. Let A be a connected subspace of X and let $A \subset B \subset \bar{A}$.

To prove B is connceted.

Suppose B is not connected. Then we can write, $B = C \cup D$ where C and D are nonempty set with $\bar{C} \cap D = C \cap \bar{D} = \varnothing$.

We have, $A \subset B$

$\Rightarrow A \subset C \cup D$.

Since A is connceted, By a theorem, $A \subset C$ or $A \subset D$.

Assume that, $A \subset C$

$\Rightarrow \bar{A} \subset \bar{C}$

$\Rightarrow B \subset \bar{C}$

$\Rightarrow B \cap D = \varnothing$.

But $B = C \cup D$. Therefore, $D = \varnothing$.

Which is a contradication to D is a nonempty set. Therefore, our assumtion is wrong. Therefore, B is connected.

Theorem 0.10.13. *The image of a connected space under a continuous map is connected.*

Proof. Let $f : X \to Y$ be a continuous map. Given X is connected.

To prove f(X) is connected.

Suppose f(X) is not connected. Then we can write, $f(X) = A \cup B$ where A and B are nonempty disjoint open set in f(x).

Let $g : X \to f(X)$ with g(x) = f(x), $\forall x \in X$. Then g is onto and continuous.

Now, $X = g^{-1}(f(x))$

$= g^{-1}(A \cup B)$

$= g^{-1}(A) \cup g^{-1}(B)$.

Since g is continuous and A and B are nonempty open set in $g^{-1}(A)$ and $g^{-1}(B)$ are open. Therefore,$g^{-1}(A)$ and $g^{-1}(B)$ are open in X.

Thus $X = g^{-1}(A) \cup g^{-1}(B)$ where $g^{-1}(A)$ and $g^{-1}(B)$ are nonempty open set with $g^{-1}(A) \cap g^{-1}(B) = \varnothing$.

Therefore, X is not connected.

Which is a contradication to X is connected. Therefore, our assumption is wrong.
Therefore, f(x) is connceted.

Theorem 0.10.14. *A finite cartesian product of connected space is connected.*

Proof. Let $X_1, X_2, \ldots, X_n$ be connected spaces.

To prove $X_1 \times X_2 \times \ldots \times X_n$ is connected.

First we prove product of two connected spaces X is connected.

Choose a base point $a \times b$ in the product $X \times Y$. Note that, the horizontal slice $X \times b$ is connected being homeomorphic with X and each vertical slice $X \times Y$ is connected being homeomorphic with Y.

For each $x \in X$, define T-shaped space, $T_x = (X \times b) \cup (x \times Y)$.

We have $x \times b \in X \times b$ and $x \times b \in x \times Y$.

Therefore, $x \times b \in (x \times b) \cap (x \times Y)$.

$\Rightarrow (x \times b) \cap (x \times Y)$ 6= $\varnothing$.

By a theorem, $x \times b \cup x \times Y$ is connected. Therefore, T_x is connected for every $x \in X$.

Claim : $X \times Y = S_x T_x$

Clearly, $T_x \subseteq X \times Y$ for every $x \in X$.

Therefore, $S_{x \in X} T_x \subseteq X \times Y$ ——————-(1).

Now, To prove $X \times Y \subseteq S_{x \in X} T_x$.

We have, $x \times y \in X \times Y$

$x \times Y \in x \times Y \subset T_x$

$x \times y \in T_x \subseteq ST_x$

$X \times Y \subseteq S_{x \in X} T_x$ ——————-(2).

From equations (1) and (2) we get, $X \times Y = S_{x \in X} T_x$.

We have $(a, b) \in X \times b$

Therefore, $(a, b) \in T_x \; \forall \; x \in X$.

Therefore, $T_{x \in X} T_x$ 6= $\varnothing$.

Thus $X \times Y = S_{x \in X} T_x$ where $T_{x \in X} T_x$ 6= $\varnothing$.

By a lemma, $X \times Y$ is connected as each T_x is connected.

Now, we prove that cross product of finite number of connected spaces is connected.
Let $X_1, X_2, \ldots, X_n$ be n-connected spaces.

To prove $X_1 \times X_2 \times \ldots \times X_n$ is connected.

By the observation, we say that $X_1 \times X_2$ is connected. Therefore, the result is

true for $n = 2$.
Assume that the result is true for n-1.
That is $X_1 \times X_2 \times . . . \times X_{n-1}$ is connected.
To prove the result is true for n.
We have, $X_1 \times X_2 \times . . . \times X_n$ is homeomorphic with $(X_1 \times X_2 \times . . . \times X_{n-1}) \times X_n$.
By our assumption, $(X_1 \times X_2 \times . . . \times X_{n-1})$ is connected. Therefore,$(X_1 \times X_2 \times . . . \times X_{n-1}) \times X_n$ is connected.
$\Rightarrow (X_1 \times X_2 \times . . . \times X_{n-1}) \times X_n$ is connected.

0.11 Compact spaces

Definition 0.11.1. *A collection* A *of subsets of X is said to be cover X or to be a covering of X if the union of elements of* A *is equal to X.*
Definition 0.11.2. *A collection* A *of open subsets of X is said to be a open covering of X if its union of elements of* A *is equal to X.*
Definition 0.11.3. *A space X is said to be compact if every open covering* A *of X contains a subcollection that also covers X.*
Example 0.11.4. *The real line R is not connected.*
Let $A = \{(n, n+2)/n \in Z\}$ *be a collection of open subsets of R whose union is R. But this collection does not have a finite subcollection that covers R.*
Example 0.11.5. *Let* $X = \{0\} \cup \{\frac{1}{n}/n \in Z_+\}$ *be a subspace of R. Then X is compact. Let* $\{U_\alpha\}$ *be an open covering of X. Therefore,* $X = \bigcup_\alpha U_\alpha$.
$0 \in X \Rightarrow 0 \in \bigcup_\alpha U_\alpha$
$\Rightarrow 0 \in U_\alpha$ for some α.
U_α *is an open set containing zero. Therefore,* U_α *is a neighbourhood of zero. Since* $\frac{1}{n} \to 0$*, there exists a positive integer N such that* $\frac{1}{n} \in U_\alpha \ \forall \ n \geq N$.
$\Rightarrow \frac{1}{N}, \frac{1}{N+1}, . . . , 0 \in U_\alpha$.
Now, $1, \frac{1}{2}, . . . , \frac{1}{N-1}$ *are in* $\bigcup U_\alpha$.
Let $1 \in U_{\alpha_1}, \frac{1}{}$

$\frac{1}{2} \in U_{\alpha_2}, \ldots, \frac{1}{N-1} \in U_{\alpha_{N-1}}$.

Therefore, $\{1, \frac{1}{2}, \ldots, \frac{1}{N-1}, \frac{1}{N}, \frac{1}{N+1}, \ldots, 0\} \subset U_{\alpha_1} \cup U_{\alpha_2} \cup \ldots \cup U_{\alpha_{N-1}} \cup U_\alpha$

$\Rightarrow X \subset U_{\alpha_1} \cup U_{\alpha_2} \cup \ldots \cup U_{\alpha_{N-1}} \cup U_\alpha$

$\Rightarrow \{U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_{N-1}}, U_\alpha\}$ *is a finite subcollection which covers X. Therefore, X is compact.*

Example 0.11.6. $(0, 1]$ *is not compact. Since the open covering* A = $\{(\frac{1}{n}, 1)/n \in \mathbb{Z}_+\}$ *contains no finite subcollection covering* $(0, 1]$

Example 0.11.7. $(0, 1]$ *is not compact and* $[0, 1]$ *is compact.*

Definition 0.11.8. *If Y is the subspace of X, a collection* A *of subset of X is said to cover Y if the union of this element contains Y.*

Lemma 0.11.9. *Let Y be a subspace of X. Then Y is compact if and only if every covering of Y bysets open in X contains a finite subcollection covering Y.*

Proof. First assume Y is compact and let A = $\{A_\alpha\}_{\alpha \in J}$ is a covering of Y bysets open in X.

Now, consider the collection $\{A_\alpha \cap Y\}_{\alpha \in J}$ this is the covering of Y bysets open in Y.

Since $A_\alpha \cap Y$ is open in Y for each α. Therefore, by compactness of Y, this collection has a finite subcollection $\{A_{\alpha_1} \cap Y, A_{\alpha_2} \cap Y, A_{\alpha_3} \cap Y, \ldots, A_{\alpha_n} \cap Y\}$ that covers Y.

Then $\{A_{\alpha_1}, A_{\alpha_2}, \ldots, A_{\alpha_n}\}$ is the finite subcollection of A that covers Y.

Conversely, assume every covering of Y bysets open in X contains a finite subcollection covering Y.

To prove Y is compact.

Let $A' = \{A'_\alpha\}$ be a covering of Y bysets open in X.

For, each α choose a set A_α open in X such that $A'_\alpha = A_\alpha \cap Y$.

$Y = A'_{\alpha_1} \cup A'_{\alpha_2} \cup \ldots \cup A'$

$\alpha_i \cup \ldots$

$Y = (A_{\alpha_1} \cap Y) \cup (A_{\alpha_2} \cap Y) \cup \ldots \cup (A_{\alpha_i} \cap Y) \ldots$

$= Y \cap (A_{\alpha_1} \cup A_{\alpha_2} \cup \ldots)$

$Y \subset A_{\alpha_1} \cup A_{\alpha_2} \cup \ldots \cup A_{\alpha_i} \cup \ldots.$

The collection $\{A_\alpha\}$ is the covering of Y bysets open in X. Therefore, by hypothesis, some finite subcollection $\{A_{\alpha_1}, A_{\alpha_2}, \ldots, A_{\alpha_n}\}$ covers Y.

Then $\{A'_{\alpha_1}, A'_{\alpha_2}, \ldots, A'_{\alpha_n}\}$ is the subcollection of A' that covers A. Therefore, Y is compact.

Theorem 0.11.10. *Every closed subsets of a cmpact space is compact.*

Proof. Given X is compact. Let Y be a closed subset of a compact set X.
To prove Y is compact.

Let $A = \{A_\alpha\}_{\alpha \in J}$ be a covering of Y bysetsopen in X.

Let us form an open covering β of Y by adjoining to A, single open set X-Y.

Since X is compact, there exists a finite subcollection $\{A_{\alpha_1} \cup A_{\alpha_2} \cup \ldots \cup A_{\alpha_n} \cup X - Y\}$ of β that covers X. Therefore, $X = \{A_{\alpha_1} \cup A_{\alpha_2} \cup \ldots \cup A_{\alpha_n} \cup X - Y\}$.

Then $Y \subset A_{\alpha_1} \cup A_{\alpha_2} \cup \ldots \cup A_{\alpha_n}$.

$\Rightarrow$ There exists a finite subcollection of A which covers Y. Therefore, by previous lemma, Y is compact.

Theorem 0.11.11. *Every compact subset of a hausdorff space is closed.*

Proof. Let X be a hausdorff space. Let Y be a compact space of X.
To prove Y is closed in X.
That is to prove X-Y is open in X.

Let $x_0 \in X - Y$

$\Rightarrow x_0 \notin Y$

Then $x_0 \neq y \ \forall y \in Y$.

Now, x_0 and y are two distinct points of Hausdorff space X.
For, each point y of Y, there exists a disjoint neighbourhood U_y and V_y of x_0 and y respectively.

Now, the collection $\{V_y / y \in Y\}$ is the collection of open in X and $Y \subset \bigcup_{y \in Y} V_y$.

Therefore, $\{V_y / y \in Y\}$ is the covering of Y bysets open in X.

By lemma, there exists a finite subcollection $\{V_{y_1}, V_{y_2}, \ldots, V_{y_n}\}$ that covers Y.

That is $Y \subset V_{y_1} \cup V_{y_2} \cup \ldots \cup V_{y_n}$.

Let $V = V_{y_1} \cup V_{y_2} \cup \ldots \cup V_{y_n}$. Then $Y \subset V$ and V is open in X.

Let $U = U_{y_1} \cap U_{y_2} \cap \ldots \cap U_{y_n}$.

Therefore, U is the finite intersection of open sets containing x_0.
Therefore, U is an open sets containing x_0.

Claim: $U \cap V = \emptyset$.

Suppose $U \cap V \neq \emptyset$. Then $z \in U \cap V$

$\Rightarrow z \in U$ and $z \in V$.

Now, $z \in U \Rightarrow z \in U_{y_i} \forall i = 1, 2, \ldots, n$.

Also $z \in V \Rightarrow z \in V_{y_i}$ for some i.

$z \in U_{y_i} \cap V_{y_i}$.

Which is a contradication to $U_{y_i} \cap V_{y_i} = \emptyset$.

Therefore, $U \cap V = \emptyset$. Also $Y \subset U$.

$\Rightarrow U \cap Y = \emptyset$

$\Rightarrow U \subset X - Y$

$\Rightarrow X - Y$ is open in X.

$\Rightarrow$ Y is closed in X.

Theorem 0.11.12. *The image of a compact space under a continuous map is compact.*

Proof. Let $f : X \to Y$ be a continuous map, where X is a compact space and Y be a topological space.

To prove $f(X)$ is compact.

Let A be a cover of $f(X)$ bysets open in Y. Then $f(X) \subset \bigcup_{A \in \mathcal{A}} A$. Since f is continuous and A is open in Y.

$\Rightarrow f^{-1}(A)$ is open in X for every $A \in \mathcal{A}$.

Also, $X = \bigcup_{A \in \mathcal{A}} f^{-1}(A)$.

Therefore, $\{f^{-1}(A)/A \in \mathcal{A}\}$ is an open covering for X.

Since X is compact, there exists a finite subcollection, $\{f^{-1}(A_1), f^{-1}(A_2), \ldots, f^{-1}(A_n)\}$ that covers X.

That is $X = f^{-1}(A_1) \cup f^{-1}(A_2) \cup \ldots \cup f^{-1}(A_n)$

$\Rightarrow f(X) \subset A_1 \cup A_2 \cup \ldots \cup A_n$.

$\{A_1, A_2, \ldots, A_n\}$ is a finite subcollection of A that covers f(X).

By a lemma, $f(X)$ is compact.

Theorem 0.11.13. *Let $f : X \to Y$ be a bijective continuous function, if X is compact and Y is hausdorff, then f is a homeomorphim.*

Proof. Let $f : X \to Y$ be a bijective continuous function. Given X is compact and Y is hausdorff.
To prove f is a homeomorphic.
It is enough to prove f^{-1} is continuous.
That is to prove that $(f^{-1})^{-1}(A)$ is closed in Y, for every closed set A in X.
Thatis, to prove f(A) is closed in Y for every closed set A in X.

Let $A \subset X$ be closed in X.

Now, A being closed subset of the compact set X, A is compact.
Now, $f(A)$ being a continuous image of a compact set A, $f(A)$ is compact.
Again, $f(A)$ being a compact subset of a hausdorff space Y.
Therefore, $f(A)$ is closed.
Therefore, f^{-1} is continuous.
Therefore, f is a homeomorphism.

Theorem 0.11.14. *The product of finitely many compact space is compact.*

Proof. Let $X_1, X_2, \ldots, X_n$ be compact spaces.

To prove $X_1 \times X_2 \times \ldots \times X_n$ is compact.

First we shall prove that the product of two compact space is compact.
Then the theorem follows by induction for any finite product.
Before proving this theorem, let us prove the Tube lemma. Consider the product space $X \times Y$ where Y is compact. If N is an open set of $X \times Y$ containing the slice $x_0 \times Y$ of $X \times Y$, then N contains some tube $W \times Y$ about $x_0 \times y$ where W is a neigbourhood of $x_0$in X.
We prove the following, there is a neighbourhood W of x_0 in X such that
$W \times Y \subset N$.

$W \times Y$ is often called a tube about $x_0 \times Y$.

First let us cover $x_0 \times Y$ by basis elements $U \times V$ (for the topology of $X \times Y$ lieing in N).

The space $x_0 \times Y$ is compact being homeomorphic to Y.

We can cover $x_0 \times Y$ by finitely many such basis element $U_1 \times V_1, U_2 \times V_2, \ldots, U_n \times V_n$.

We assume that each basis element $U_i \times V_i$ intersects $x_0 \times Y$.
Since otherwise the basis element would be super fluous we can discard it forms the finite collection and still the covering of $x_0 \times Y$.

Define $W = U_1 \cap U_2 \cap \ldots \cap U_n$.

Then the set W is open and it contains x_0 because each $U_i \times V_i$ intersects $x_0 \times Y$.

we assume that the sets $U_i \times V_i$ which were choose to cover $x_0 \times Y$ actually cover the tube $W \times Y$.

For, let $X \times y \in W \times Y$.

Consider the point $x_0 \times y$ of the slice $x_0 \times Y$, having the same y-coordinate at this point.

Now, $x_0 \times y \in U_i \times V_i$ for some i.

So that $y \in V_i$.

But $x \in U_j$ for all j.

We have $x \times y \in U_i \times V_i$. Therefore, $W \times Y \subset N$. Hence the lemma.

Proof of the main theorem:

Let X and Y be compact space.

To prove $X \times Y$ is compact.

Let A be an open covering of $X \times Y$.

Given $x_0 \in X$, the slice $x_0 \times Y$ is compact and therefore it can be covered by finitely many elements $A_1, A_2, \ldots, A_m$ of A .

Their union $N = A_1 \cup A_2 \cup \ldots \cup A_m$ is an open set containing $x_0 \times Y$.

By above tube lemma, the open set N contains a tube $W \times Y$ about $x_0 \times Y$, where W is open in X.

Then $W \times Y$ is covered by finitely many elements $A_1, A_2, \ldots, A_m$ of A .

Thus for each $x \in X$, we can choose a neigbourhood W_x of X such that the tube $W_x \times Y$ can be covered by finitely many elements of A .

Since X is compact. There exists a finite subcollection $\{W_1, W_2, \ldots, W_k\}$ which covers X.

Therefore, theunion of the tubes $W_1 \times Y, W_2 \times Y, \ldots, W_k \times Y$ covers all of $X \times Y$.

Since each may be covered by finitely many elements of A .

Hence $X \times Y$ has a finite subcover. Thus $X \times Y$ is compact.

By induction, it follows that $X_1, X_2, \ldots, X_n$ are compact spaces then their product $X_1 \times X_2 \times \ldots \times X_n$ is compact.

Definition 0.11.15. *A collection* C *of subsets of X is said to satisfy the finite intersection properly if for every finite subclooection* $\{C_1, C_2, \ldots, C_n\}$ *of* C *, the intersection* $C_1 \cap C_2 \cap \ldots \cap C_n$ *is nonempty.*

Theorem 0.11.16. *Let X be a tropological space. Then X is compact if and only if for every collection* C *of closed sets in X having the finite intersection property, the intersection* $\bigcap_{C \in \mathcal{C}} C$ *of all the elements of* C *is nonempty.*

Proof. Suppose X is compact.

Let C be a collection of closed sets in X satisfiying the finite intersection condition.

To prove $\bigcap_{C \in \mathcal{C}} C \neq \varnothing$.

If not assume, $\bigcap_{C\in\mathcal{C}} C = \emptyset$.

Then $X - \bigcap_{C\in\mathcal{C}} C = X - \emptyset$.

Since C is closed for all $C \in \mathcal{C}$, X- C is open for all $C \in \mathcal{C}$. Therefore, $\{X - C / c \in \mathcal{C}\}$ is a collection of open subsets of X and $X = \bigcap_{C\in\mathcal{C}} (X - C)$.

Therefore, $\{X - C/C \in \}$ is an open cover for X. Since X is compact, there exists a finite subcollection, $\{X - C_1, X - C_2, \ldots, X - C_n\}$ which covers X.

Therefore, $X = (X - C_1) \cup (X - C_2) \cup \ldots \cup (X - C_n)$

$\Rightarrow X = X - (C_1 \cap C_2 \cap \ldots \cap C_n)$

$\Rightarrow C_1 \cap C_2 \cap \ldots \cap C_n = \emptyset$.

Which is a contradication to $\mathcal{C}$ satisfies the finite intersection condition, $\bigcap_{C\in\mathcal{C}} C \neq \emptyset$.

Conversely, suppose that for every collection $\mathcal{C}$ of closed sets in X having the finite intersection property, the intersection $\bigcap_{C\in\mathcal{C}} C$ of all elements of $\mathcal{C}$ is nonempty.

To prove X is compact.

Suppose X is not compact.

Then there exists an open covering $\mathcal{A}$ for X which contains no finite subcovering.

Since $\mathcal{A}$ is an open covering for X.

$X = \bigcup_{A\in\mathcal{A}} A$. Then $X - X = X - \bigcup_{A\in\mathcal{A}} A$.

That is $\emptyset = \bigcap_{A\in\mathcal{A}} (X - A)$ —————-(1)

Now, $\{X - A/A \in \mathcal{A}\}$ is a collection of closed sets in X.

Let $\{X - A_1, X - A_2, \ldots, X - A_n\}$ be a subcollection of $\{X - A/A \in \mathcal{A}\}$.

Then $(X - A_1) \cap (X - A_2) \cap \ldots \cap (X - A_n) = X - (A_1 \cup A_2 \cup \ldots \cup A_n) \neq \emptyset$.

Therefore, $\{X - A/A \in \mathcal{A}\}$ is a collection of closed subsets of X satisfying the finite intersection condition and by (1) $\bigcap_{A\in\mathcal{A}} (X - A) = \emptyset$.

Which is a contradication.

Therefore, our assumption is wrong.

Hence X is compact.

Unit – V

(1)

Limit Point Compactness

Definition :-

A space X is said to be limit point compact if every infinite subset of X has a limit point.

Theorem :-

Compactness implies limit point compactness, but not conversely.

Proof :

Let X be compact space.

Given a subset A of X,

To prove : A is infinite then A has a limit point.

We prove the contrapositive.

If A has no limit point, then A must be finite.

Suppose A has no limit point.

Then A contains all its limit points, so that A is closed.

Furthermore, for each $a \in A$ we can choose a neighbourhood U_a of a such that U_a intersects A in the point a alone.

The space X is covered by the open set X − A and the open sets U_a;

X being compact it can be covered by finitely many of these sets.

Since X − A does not intersect A, and each set U_a contains only one point of A the set A must be finite.

Hence the proof.

Definition:

Let X be a topological space. If (x_n) is a sequence of points of X, and if $n_1 < n_2 < \cdots < n_i < \cdots$ is an increasing sequence of positive integers, then the sequence (y_i) defined by setting $y_i = x_{n_i}$ is called a subsequence of the sequence (x_n). The space X is said to be sequentially compact if every sequence of points of X has a convergent subsequence.

Theorem:-

Let X be a metrizable space. Then the following are equivalent.

1). X is compact.

2) X is limit point compact.

3) X is sequentially compact.

Proof:

① ⇒ ② already proved. (Theorem ①)

To prove:- ② ⇒ ③

Assume X is limit point compact.

Claim: X is sequentially compact.

Given a sequence (x_n) of points of X, consider the set

$A = \{x_n \mid n \in \mathbb{Z}_+\}$

If the set A is finite, then there is a point x such $x = x_n$ for infinitely many value of n. 3

Then the sequence (x_n) has a subsequence that is constant and therefore converses trivially.

On the other hand, if A is infinite, then A has a limit point.

We define a subsequence of (x_n) converges to x as follows:

First choose n_1 so that

$$x_{n_1} \in B(0, 1).$$

Suppose that the positive integer n_{i-1} is given.

Because the ball $B(x, 1/i)$ intersects A in infinitely many points, we can choose an index $n_i > n_{i-1}$ such that

$$x_{n_i} \in B(x, 1/i).$$

Then the subsequence $x_{n_1}, x_{n_2}, \ldots$ converges to x.

To prove:- (3) ⟹ (1)

First, we show that if X is sequentially compact, then the lebesgue number lemma:- holds for X.

Let A be an open covering of X.

We assume that there is no $\delta > 0$ such that each set of diameter less than δ has an element of containing it, and derive a contradiction.

By our assumption, for each positive integer n, there exists a set

4 of diameter less than $1/n$ that is not contained in any element of $\mathcal{A}$.

Let C_n be such a set.

Choose a point $x_n \in C_n$ for each n.

By hypothesis, some subsequence (x_{n_i}) of the sequence (x_n) converges, say to the point a.

Now, a belongs to some element A of the collection $\mathcal{A}$.

Because A is open, we may choose an $\varepsilon > 0$ such that $B(a, \varepsilon) \subset A$.

If i is large enough that $1/n_i < \varepsilon/2$, then the set C_{n_i} lies in the $\varepsilon/2$ neighbourhood of x_{n_i}.

If i also chosen large enough that $d(x_{n_i}, a) < \varepsilon/2$, then C_{n_i} lies in the ε-neighbourhood of a.

This means that $C_{n_i} \subset A$, contrary to hypothesis.

$\therefore$ The lebesque lemma holds for X.

To show that if X is sequentally compact, then given $\varepsilon > 0$, there exist a finite covering of X by open ε-balls.

Once again there exists an $\varepsilon > 0$ such that X cannot be covered by finitely many ε-balls.

Construct a sequence of points x_n of X as follows:

First, choose x_1 to be any point of X.

$B(x_1, \varepsilon)$ is not all of X (otherwise X could be covered by a single ε-ball).

Choose x_2 to be a point of X not in $B(x_1, \varepsilon)$.

In general, given $x_1, \ldots, x_n$, choose x_{n+1} to be a point not in the union $B(x_1, \varepsilon) \cup \ldots \cup B(x_n, \varepsilon)$.

Using the fact that these balls do not cover X.

By our construction $d(x_{n+1}, x_i) \geq \varepsilon$ for $i = 1, 2, \ldots, n$.

$\therefore$ the sequence (x_n) can have not convergent subsequence; infact, any ball of radius $\varepsilon/2$ can contain x_n for at next one value of n.

Finally, we show that if X is sequentally compact, then X is compact.

Let A be an open covers of X.

Because X is sequentially compact, the open covering A has a lebesque number δ.

Let $\varepsilon = \delta/3$; use sequentially compactness of X to find a finite covering of X by open ε-balls.

Each of these balls has diameter atmost $2\delta/3$. element of A.

So it lies an element of A.

Choosing one such element of A for each of the ε-balls, we obtain a finite subcollection of A that

Covers X.

Local Compactness

Definition:

A space X is said to be locally compact at x if there is some compact subspace C of X that contains a neighbourhood of x. If X is locally compact at each of sets points, X is said simply to be locally compact.

Note:

Every compact space is locally compact.

Ex:-1

The real line $\mathbb{R}$ is locally compact.

The point x lies in some interval (a, b), which is turn is contained in the compact subspace [a, b].

The subspace $\mathbb{Q}$ of rational numbers is not locally compact.

Ex:-2 The space $\mathbb{R}^n$ is locally compact.

The point x lies in some basis element $(a_1, b_1) \times \cdots \times (a_n, b_n)$ which in term lies compact subspace $[a_1, b_1] \times \cdots \times [a_n, b_n]$.

The space $\mathbb{R}^\omega$ is not locally compact.

None of its basis elements are contained in compact subspaces.

For, if

$B = (a_1, b_1) \times \dots \times (a_n, b_n) \times \mathbb{R} \times \mathbb{R} \times \dots$

where contained in a compact subspace, then its closure

$\overline{B} = [a_1, b_1] \times \dots \times [a_n, b_n] \times \mathbb{R} \times \dots$

would become part, which is not true.

X → Simply order set → locally compact

Definition:

If Y is a compact Hausdorff space and X is a proper subspace of Y whose closure equals Y, then Y is said to be compactification of X. If Y − X equals a single point, then Y is called the one-point compactication of X.

R → ℂ ∪ {∞} → Rieman sphere (extended complex plane) (complex number)

R → circle

$R^2 \to S^2$ sphere

Theorem: 29.1

Let X be a space. Then X is locally compact Hausdorff if and only if there exists a space Y satisfying the following conditions.

(3)

1). X is subspace of Y.

2). The set Y − X consists of a single point

Proof :-

Step :- 1

we first verify uniqueness.

Let Y and Y' be has spaces satisfying the given three conditions.

Define $h : Y \to Y'$ by letting h map the single point p of $Y - X$ to the point q of $Y' - X$, and letting h equal the identity on X.

$\therefore h$ is bijection (h equals the identity on X).

To prove :- h is homeomorphism.

It is enough to prove that $h(U)$ is open in Y' if U is open in Y.

Case (i) :- U does not contain p.

Then $h(U) = U$.

Since U is open in Y and $U \subset X$, it is open X.

Because $Y' - X = \{q\}$ which is closed. X is open in Y'.

$\therefore$ the set U is open in Y'.

(By lemma) for

$\therefore h(U)$ is open in Y'.

$\therefore h$ is a homeomorphism and Y is unique.

Case (ii) :- U containing p.

Let $C = Y - U$.

then C, is closed in Y.

Also C is a closed subset of a compact space Y.

$\therefore C$ is a compact subspace of X.

Because X is a subspace of Y', C is also a compact subspace of Y'.

now $U = Y - C$.

$h(U) = h(Y - C) = Y' - C$

since C is compact subspace of hausdorff space Y', C is closed in Y'.

$\therefore h(U) = Y' - C$ is open in Y'.

$\therefore h$ is a homeomorphism and hence Y is unique.

Step :- 2

Now we suppose X is locally compact Hausdorff and construct the space Y.

Let $Y = \{\infty\} \cup X$, where ∞ is some set a point of X.

Consider the collecting $\mathcal{T}$ of subsets of Y of the type ① all set U that are open in X and ② all sets of the form $Y - C$, where C is a compact subspace of X.

prove $\mathcal{T}$ is a topology on Y.

The empty set is of type ① and the space Y is of type ②.

Intersection of two open sets is open. Involves three cases;

$U_1 \cap U_2 \longrightarrow$ is of type ①

$(Y - C_1) \cap (Y - C_2) = Y - (C_1 \cup C_2)$ is of type ②

$U_1 \cap (Y - C_1) = U_1 \cap (X - C_1)$ is of type ①.

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Because U_1 is open X and C_1 is Compact subset of a Hausdorff space Y, C_1 is closed in X.

$\Rightarrow X - C_1$ is open in X

$\Rightarrow U \cap (X - C_1)$ is open in $\mathcal{T}$!

$\therefore$ The intersection of two open sets is open.

To prove, the union of any collection of open sets is open.

there are three cases;

$\cup U_\alpha = U$ is of type ①

$\cup (Y - C_\beta) = Y - (\cap C_\beta) = Y - C$ is of type ②

$(\cup U_\alpha) \cup (\cup (Y - C_\beta)) = U \cup (Y - C)$.
$= Y - (C - U)$

which is of type ②. Because $C - U$ is a closed subspace of C and is Compact.

$\therefore$ Arbitrary union of open sets is open.

Hence $\mathcal{T}$ is a topology on Y.

we show that X is a subspace of Y.

The open set in the subspace topology are of the form $X \cap U$ where U is open in Y.

$\therefore$ If U is of type ①, the $U \cap X = U$ is open in X.

If $Y - C$ is of type ②, then $(Y - C) \cap X = X - C$ is open in X.

Conversely,

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If $U \subset X$ is open, then U is open in Y of type ①.

$\because X$ is a subspace of Y.

To show that Y is compact.

Let A be an open covering of Y.

Some element $V \in A$ must contain $\infty \notin X$.

Hence V is of the form $V = Y - C$.

The collection A of open subsets of Y, covers the compact space C.

So, there is a finite subcollection $\{U_1, U_2, \ldots, U_n\} \subset A$ that covers C.

Then $\{U, U_1, U_2, \ldots\}$ is a finite subcover of Y.

$\therefore Y$ is compact.

<u>To prove</u>: Y is Hausdroff space.

Let $x, y \in Y$.

If both lie in X, then there are disjoint open subsets U and V of X with $x \in U$ and $y \in V$ {$\because X$ is hausdroff}

$\therefore U$ and V are disjoint open sets of Y.

$\therefore Y$ is hausdroff space.

On the other hand, if $x \in X$ and $y = \infty$, we can choose a compact subset C in X containing a neighbourhood U of x.

Then $x \in U$ and $V = Y - C$ are disjoint neighbourhood of x and ∞ respectively in Y.

$\therefore$ Y is Hausdroff space.

<u>Step:- 3</u>

Conversely, suppose Y is a space satisfying conditions ① – ③ exists.

<u>To prove</u>:- X is locally compact hausdroff space.

Since X is a subspace of the hausdroff space Y, X is hausdroff.

<u>To prove</u>:- X is locally compact.

Let $x \in X$.

Let y be a single point of $Y - X$.

Since Y is hausdroff, there exist disjoint open sets U and V with $x \in U$ and $y \in V$.

Let $C = Y - V$.

Then C is closed in Y.

i.e), C is a closed subset of a compact subspace Y.

$\therefore$ C is compact.

Since C lies in X, it is also compact subspace of X.

It contains the neighbourhood U of x.

i.e), $x \in U \subset C \subset X$.

$\therefore$ X is locally compact at x.

Since $x \in X$ is arbitrary, X is locally compact at point of X.

$\therefore$ X is locally compact.

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Theorem :-

Let X be a Hausdroff space. Then X is locally compact if and only if given x in X, and given a neighbourhood U of x, there is a neighbourhood V of x such that $\bar{V}$ is Compact and $\bar{V} \subset U$.

Proof :-

Suppose given $x \in X$ and given a neighbourhood U of x, there is a neighbourhood V of x such that $\bar{V}$ is Compact and $\bar{V} \subset U$.

i.e), $x \in V \subset \bar{V} \subset U$.

$\therefore \bar{V}$ is a compact space containing the neighbourhood of U of x.

$\therefore X$ is locally compact.

Conversely,

Assume that X locally compact and Hausdroff space.

Let x be a point of X and U be a neighbourhood of x.

Let $Y = X \cup (\infty)$ be the one point Compactification Y of X. and let $C = Y - U$

Then ~~C~~ $C \subset Y$ the closed in Y and C is compact subspace of Y.

Hence C is compact.

Since Y is Hausdroff space and $x \notin C$ we can find disjoint open subsets V, W Containing x and C and $V \cap W \neq \phi$,

Then the $\bar{V}$ of V in Y is Compact.

Further $\bar{V}$ is ~~sub~~ disjoint from C

to that $\bar{V} \subset U$.

Hence the proof.

Corollary:- ①

Let X be locally compact Hausdorff. Let A be a subspace of X. If A is closed in X or open in X, then A is locally compact.

proof:- Suppose that A is closed in X.

Given $x \in A$, let C be a compact subspace of X containing the neighbourhood U of x in X.

Then $C \cap A$ is closed in C and thus compact and it is contains the neighbourhood $U \cap A$ of x in A.

$\therefore A$ is locally compact.

Suppose A is open in X.

Given $x \in A$, by theorem, there exists a neighbourhood of U of x in X such that $\bar{V}$ is compact and $\bar{V} \subset A$.

Then $C = \bar{V}$ is a compact subspace of A containing the neighbourhood V of x on A.

$\therefore A$ is locally compact.

A space X is homeomorphic to an open subspace of a compact Hausdorff space locally compact Hausdorff. proof:- Corollary ① and then them (29.1)